DOCTORAL THESIS:
Canonical representations of Gaussian processes and their applications to interpolation problem

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Chapter 0

Introduction

The canonical representation theory of Gaussian processes has been originated by Lévy [17] and systematically developed by Hida [9] and Cramér. The idea of this representation is to express a given Gaussian process in terms of Brownian motion, the properties of which are well known, and a nonrandom kernel function, so that probabilistic structure of the process is represented in a visualized manner.

Let us give precise formulation. Suppose a Gaussian process $X = \{X(t) ; t \in (t_0, t_1)\}$ is expressed in the form

$$X(t) = \int_{t_0}^{t} F(t, u) dB(u), \text{ for any } t \in (t_0, t_1),$$

by the use of a process $B = \{B(t) ; t \in (t_0, t_1)\}$ with independent increments. Here the nonrandom function $F(t, \cdot)$ is square integrable with respect to $dv(t) = E[dB(t)^2]$. In this case, $X$ is said to be represented by $(F, B)$. Furthermore, if the closed linear manifold $\mathcal{M}_t(X)$ spanned by $\{X(s); s \leq t\}$ is identical with $\mathcal{M}_t(B)$, for any $t \in (t_0, t_1)$, then $X$ is said to be canonically represented by $(F, B)$. This means that the past informations of $X$ and of $B$ are identical in each time $t \in (t_0, t_1)$. The canonical representation of a Gaussian process is our main concern.
in the present paper. The representation has been very convenient to the prediction problem. We shall make progress in the canonical representation theory, and apply it to the interpolation problem as well as to the prediction problem.

The purpose of Chapter 1 is to introduce a new concept of a backward representation of Gaussian processes, for which time development is observed backward. The existing theory of representation may now be called the forward representation. Combining both backward and forward representations we are able to find a profound structure of the given Gaussian process. In the next chapter, by applying the theory of backward representation, the interpolation problem for multiple Markov Gaussian processes is discussed. Lévy got a motivation of canonical representation from a Gaussian process $M_d(t)$, which is the spherical mean of the $d$-dimensional parameter Brownian motion originated by himself. The remarkable property of the process $M_d$ is a multiple Markov property for odd $d$. However, it has no Markov property at all for even $d$. So a multiple Markov process plays an important role and gives a typical example in the canonical representation theory. Some properties of such processes have been investigated by many authors like Hida [9], Mandrekar [19] and Pitt [22]; now we go one step further to find some more interesting properties.

For these purposes, we first introduce the backward canonical representation for a Gaussian process in Section 1.1, having been motivated by Lévy's work. It is convenient to observe representations of stationary processes before we come to the main topic, since the spectral analysis for such processes is well established in an elementary manner and ready to be used in our case (Section 1.2).

Section 1.3 is devoted to a multiple Markov Gaussian process. In this case the forward canonical representation does not always have unit multiplicity, nor does the backward one. This fact causes some difficulty. The so-called Goursat representation plays an important role in our approach where we have characterized both representations. The Goursat representation has been introduced by Mandrekar [19]. It expresses the $N$-ple Markov Gaussian process $X$ as the
scalar product of a nondegenerate $N$ variates martingale $(U_1, U_2, \ldots, U_N)$ and an $N$-dimensional nonrandom function $\{f_1, f_2, \ldots, f_N\}$ in the form

$$X(t) = \sum_{i=1}^{N} f_i(t)U_i(t).$$

Analogously to the canonical representation, the Goursat representation is called a proper Goursat representation if $\mathcal{M}_t(X) = \mathcal{M}_t(U_1, \ldots, U_N)$, for any $t \in (t_0, t_1)$. It is known that an $N$-ple Markov Gaussian process has the proper Goursat representation and that the representation is essentially unique. Conversely, if a process has the proper Goursat representation, it is a multiple Markov process. And moreover, the generalized canonical representation with multiplicity $M \leq N$ can be constructed from the proper Goursat representation [22]. Our main result says that the backward and the forward proper Goursat representations are completely characterized by the use of the maximal and minimal solutions for an equation derived from the covariance function of the multiple Markov Gaussian process (Theorem 1.6).

If we assume a certain analytic property in order to have a multiple Markov Gaussian process in the restricted sense, we can prove unit multiplicity and a more explicit expression is obtained for the process in question (Section 1.4).

In Chapter 2, the interpolation of a Gaussian process will be discussed with the help of the forward and the backward canonical representation. This result is far from the general theory, however it suggests to us future questions in this direction.

It is known that a reciprocal process has a simple structure for the interpolation of the process. A reciprocal process has been studied by many authors like Jamison [13], Carmichael-Massé-Theodorescu [2] and Chay [3]. It is noted that Golosov [5] obtained a general form of a reciprocal Gaussian process, and Molchan [20] refined the result. Their assertion shows that the class of the reciprocal processes is wider than that of Markov processes.
Meanwhile Hida has defined the multiple Markov property for a Gaussian process as a generalization of the ordinary (simple) Markov property in [9]. Following up Hida’s idea, Kubo [16] pointed out that Lévy’s Brownian motion \( B = \{B(x); x \in \mathbb{R}^{2p-1}\} \) has a \( p \)-ple Markov property in his sense, namely, for any \( 0 < s_1 < \cdots < s_p \), there exists a random variable \( U(x; s_1, \ldots, s_p) \), which is \( \sigma \{B(y); |y| = s_1, s_2, \ldots, s_p\} \)-measurable, satisfying that \( B(x) - U(x; s_1, \ldots, s_p) \) is independent of \( \{B(z); |z| > s_p\} \). We intend to define multiple reciprocal property analogous to the idea. As a result, the definition is a generalization of the reciprocal property.

What are discussed in Chapter 3 are as follows. In Section 3.1, we give a definition of a multiple reciprocal property as a one-dimensional parameter version of the multiple Markov property. In Section 3.2, a necessary and sufficient condition is derived for an \( N \)-ple Markov process to have the \( L \)-ple reciprocal property in terms of its covariance function.

We investigate, in Chapter 2, the interpolation problem for multiple Markov processes by using the notion of the canonical representation. In fact, we are in line with the main approach to obtain the canonical representation, when the covariance function is given. Hence, we are naturally led to a question asking what kind of effect one can expect for the canonical representation under the variation of a covariance function. It is expected that the present approach may be a first step to this problem.

In Chapter 4, we discuss the noncanonical representation in order to clarify the subtlety of the canonical property. As we see in Section 1.2, the canonical representation theory of stationary Gaussian processes is well established by Karhunen’s theorem [14]. So, we can answer what type of the noncanonical property would be possible. In Section 4.1, more detailed discussion on the canonical representations for a stationary process \( Y = \{Y(t); t \in \mathbb{R}\} \) is stated. We see that the noncanonical property of the stationary process arises from three types of reasons. Among them,
we attend to the type arising from a zero-point of the inverse Fourier transform $c(\lambda)$ of the kernel function $F$, and we discuss a relation between the zero-points of $c(\lambda)$ in the upper half-plane and the orthogonal complement of $M_t(Y)$ in $M_t(B)$. In Section 4.2, a necessary condition is presented for a stationary process in a form of some type of the innovation theorem to have an infinite dimensional orthogonal complement. In Section 4.3, we are able to obtain a systematic structure for the noncanonical representations, though Lévy [18] presented such representations for a Brownian motion in a simple method. The results in Chapter 4 are parts of a co-work with Professor M.Hitsuda and Dr. H.Muraoka.

Throughout the present paper, we assume that all the processes are mean-
continuous centered Gaussian for the sake of simplicity.

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Chapter 1

Backward canonical representations

1.1 Forward and backward canonical representations

Consider a real valued centered Gaussian process \( X = \{X(t); t \in (t_0, t_1)\} \). We start with a short review of the known result related to the canonical representation for a Gaussian process according to Hida [9]. Denote by \( \mathcal{M}_t(X) \) (resp. \( \mathcal{M}^t(X) \)) the closed linear manifold spanned by \( \{X(s); s \leq t\} \) (resp. \( \{X(s); s \geq t\} \)) in the \( L^2 \)-topology. In order to exclude the discrete multiplicity, we assume the following conditions

\((M1)\) \( \mathcal{M}_t(X) \) is continuous and increasing in \( t \),

\((M2)\) \( X \) is purely nondeterministic, that is \( \bigcap_{t \in (t_0, t_1)} \mathcal{M}_t(X) = \{0\} \),

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(M1*) \( \mathcal{M}^t(X) \) is continuous and decreasing in \( t \), and,

\[ (M2^*) \bigcap_{t \in (t_0, t_1)} \mathcal{M}^t(X) = \{0\}. \]

**Remark 1.1** The conditions \((M1)\) and \((M2)\) do not imply \((M1^*)\) and \((M2^*)\); for example, a Gaussian process \( X = \{X(t); t \in (0, 1)\} \) defined by

\[ X(t) = \begin{cases} B(t), & 0 < t < 1/2, \\ B(1/4), & 1/2 \leq t < 1, \end{cases} \]

where \( B \) is a Brownian motion, satisfies the conditions \((M1)\) and \((M2)\) but neither \((M1^*)\) nor \((M2^*)\).

**Definition 1.1** Let \( X = \{X(t); t \in (t_0, t_1)\} \) be a Gaussian process.

(i) If there exist an additive Gaussian process \( B \) having a spectral measure \( dv(t) = E[ dB(t)^2] \) and a function \( F(t, \cdot) \) of \( L^2((t_0, t), dv) \), \( t \in (t_0, t_1) \), such that \( X(t) \) is expressed in the form

\[ X(t) = \int_{t_0}^{t} F(t, u) dB(u), \tag{1.1} \]

then (1.1) is called the representation of \( X \).

(ii) The representation (1.1) is called the **(forward) canonical representation**, if it satisfies the condition

\[ \mathcal{M}_t(X) = \mathcal{M}_t(\hat{B}), \quad t \in (t_0, t_1), \tag{1.2} \]

where \( \mathcal{M}_t(\hat{B}) \) is the closed linear manifold spanned by \( \{B(s) - B(t_0); s \leq t\} \).

**Remark 1.2**

(i) It is well known that

\[ \mathcal{M}_t(\hat{B}) = \left\{ \int_{t_0}^{t} \varphi(u) dB(u); \varphi \in L^2((t_0, t)) \right\}. \tag{1.3} \]
(ii) There are in general infinitely many representations of the form (1.1); among
others, the canonical representation is unique; that is to say, if there exists
another canonical representation

\[ X(t) = \int_{t_0}^t \tilde{F}(t, u) d\tilde{B}(u), \text{ where } E[d\tilde{B}(u)^2] = d\tilde{v}(u), \]

then

\[ \int_A F(t, u)^2 d\nu(u) = \int_A \tilde{F}(t, u)^2 d\tilde{v}(u) \]

holds for any Borel set \( A \subset (t_0, t_1) \).

(iii) The property (1.2) is called proper in Hida’s paper [9].

Regarding the existence of the canonical representation, the following theorem
is well-known.

**Theorem 1.1** [9, Theorem I.5] A Gaussian process \( X = \{X(t); t \in (t_0, t_1)\} \)
satisfying (M1) and (M2) has the representation

\[ X(t) = \sum_{n=1}^M \int_{t_0}^t F_n(t, u) dB_n(u), \quad (1.4) \]

where \( B_n \)'s, \( n = 1, 2, \ldots, M \leq \infty \), are additive Gaussian processes,

\[ E[dB_m(u)dB_n(u)] = \delta_{mn} d\nu_n(u), \quad 1 \leq m, n \leq M, \]

\( d\nu_{n+1}(u) \ll d\nu_n(u) \) (absolutely continuous), \( n = 1, 2, \ldots, M - 1, \)

\( F_n(t, \cdot) \in L^2((t_0, t), d\nu_n) \),

with \( M_t(X) = \bigoplus_{n=1}^M M_t(\tilde{B}_n) \) for every \( t \in (t_0, t_1) \).
Remark 1.3 The representation (1.4) is called the \textit{generalized canonical representation} according to Hida [9]. It is the canonical representation if the multiplicity $M = 1$.

The backward canonical representation is defined analogously.

Definition 1.2 For a Gaussian process $X$, if there exist an additive Gaussian process $B^*$ having a spectral measure $d
u^*(t) = E[dB^*(t)^2]$ and a function $F^*(t, \cdot)$ of $L^2((t, t_1), d\nu^*)$, $t \in (t_0, t_1)$, satisfying

$$X(t) = \int_t^{t_1} F^*(t, u)dB^*(u), \quad (1.5)$$

and the condition

$$\mathcal{M}^t(X) = \mathcal{M}^t(\hat{B}^*), \quad t \in (t_0, t_1), \quad (1.6)$$

where $\mathcal{M}^t(\hat{B}^*)$ is the closed linear manifold spanned by $\{B^*(t_1) - B^*(s); s \geq t\}$, then $X$ is said to have the \textit{backward canonical representation} (cf. Lévy [17]).

There exists the \textit{generalized backward canonical representation} of a Gaussian process $X$ under the conditions $(M1^*)$ and $(M2^*)$. This fact can be proved in an analogous manner to Theorem 1.1.

1.2 Stationary processes

The forward canonical representation theory for a stationary Gaussian process has been established and well known. With the help of this theory we can discuss the backward representation as is briefly stated below.

Theorem 1.2 [14, Satz 5] Let $Y = \{Y(t); t \in \mathbb{R}\}$ be a mean continuous stationary Gaussian process. Then the following assertions are equivalent:

(i) The condition $(M2)$ holds.
(ii) Denote by \( f(\lambda) \) the spectral density function of \( Y \),

\[
\int_{-\infty}^{\infty} \frac{\log f(\lambda)}{1 + \lambda^2} d\lambda > -\infty.
\]  

(1.7)

(iii) \( Y \) has the forward canonical representation

\[
Y(t) = \int_{-\infty}^{t} F(t - u) dB(u), \text{ where } E[dB(u)^2] = du.
\]  

(1.8)

The representation (1.8) can be constructed as follows:

It is known that the function

\[
c(\lambda) = \sqrt{2\pi} \exp \left\{ -\frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{1 + w \lambda \log f(w)}{w - \lambda} \frac{1 + w^2}{dw} dw \right\}
\]  

(1.9)

belongs to the Hardy class \( H_2 \) in the lower half plane, and that it satisfies the relation

\[
\frac{1}{2\pi} |c(\lambda)|^2 = f(\lambda), \quad \lambda \in \mathbb{R}.
\]

And, furthermore, the Fourier transform \( F(u) \) of \( c(\lambda) \) vanishes for \( u < 0 \) by the Paley-Wiener theorem. Therefore there exists a standard Brownian motion \( B \) such that (1.8) becomes the forward canonical representation. (More precise discussion on the canonical representation of a stationary process is referred to Chapter 4.)

**Theorem 1.3** Let \( Y = \{Y(t); t \in \mathbb{R}\} \) be a mean continuous stationary Gaussian process.

(i) The condition \((M2^*)\) for \( Y \) is equivalent to \((M2)\).
(ii) Under the condition above, \( Y \) has the backward canonical representation

\[
Y(t) = \int_{t}^{\infty} F^*(t - u) dB^*(u), \text{ where } E[dB^*(u)^2] = du.
\]

(1.10)

**Proof:**

(i) The condition \((M2^*)\) comes from the spectral property like (1.7) which is invariant under the time reflection.

(ii) Put \( c^*(\lambda) = \overline{c(\lambda)} \), then \( c^*(\lambda) \) belongs to the Hardy class in the upper half plane. The Fourier transform \( F^*(u) \) of \( c^*(\lambda) \) gives the backward canonical kernel by an analogous argument to Theorem 1.2.

According to Hida-Streit [11, pp.29-30], using the time reflection operator \( T \) defined by

\[
T1 = 1 \text{ and } TY(t)T^{-1} = Y(-t),
\]

we can obtain the following result.

**Corollary 1.1** For a mean continuous stationary Gaussian process \( Y = \{Y(t); t \in \mathbb{R}\} \), the forward canonical representation (1.8), and the backward one (1.10) are connected in such a way that

\[
\begin{cases}
F^*(u) = F(-u) \\
T \hat{B}^*(t) = \hat{B}(t).
\end{cases}
\]

Applying Corollary 1.1, we obtain the backward canonical representation for a process transformable into a stationary process by a change of time scale, as follows.
Corollary 1.2  If a Gaussian process $X = \{X(t); t \in (0, \infty)\}$ has the forward canonical representation

$$X(t) = \int_0^t F(t, u) dB(u),$$

where the kernel $F(\cdot, \cdot)$ is a homogeneous function of degree $\alpha$ and $B$ is a standard Brownian motion, then the backward canonical kernel $F^*(\cdot, \cdot)$ is given by

$$F^*(t, u) = F\left(t, \frac{t^2}{u}\right) \frac{t}{u}, \quad t < u.$$

Proof: According to Lévy [17], $X$ can be transformed into a stationary process $Y = \{Y(t); t \in \mathbb{R}\}$ defined by

$$Y(t) \equiv \frac{1}{\sqrt{2}} e^{-(2\alpha+1)t} X(e^{2t}). \tag{1.11}$$

$$= \int_{-\infty}^t F\left(1, e^{-2(t-u)}\right) e^{-(t-u)} dB_0(u),$$

where $dB_0(u) = \frac{1}{\sqrt{2}e^u} dB(e^{2u})$. By Corollary 1.1,

$$Y(t) = \int_t^{\infty} F\left(1, e^{2(t-u)}\right) e^{t-u} dB_0^*(u)$$

is the backward canonical representation of $Y$, which gives us the backward canonical representation of $X$

$$X(t) = \sqrt{2t^{\alpha+\frac{1}{2}}} Y(\log \sqrt{t}) = \int_t^{\infty} F\left(t, \frac{t^2}{u}\right) \frac{t}{u} dB^*(u).$$
1.3 Multiple Markov Gaussian processes

Hida [9] has defined a multiple Markov Gaussian process in the following manner.

**Definition 1.3** [9] A Gaussian process \( X = \{ X(t); t \in (t_0, t_1) \} \) is said to be \( N \)-ple Markov, if \( \{ E[X(s_i)|\mathcal{M}_{s_0}(X)], i = 1, 2, \ldots, N \} \) is linearly independent for any \( s_0 \leq s_1 < \cdots < s_N \) and \( \{ E[X(s_i)|\mathcal{M}_{s_0}(X)], i = 1, 2, \ldots, N, N + 1 \} \) is linearly dependent for any \( s_0 \leq s_1 < \cdots < s_N < s_{N+1} \).

From the definition above, a Goursat representation has been given in the following.

**Theorem 1.4** [19, 22] A Gaussian process \( X = \{ X(t); t \in (t_0, t_1) \} \) is \( N \)-ple Markov if and only if there exist a Tchebycheff system \( \{ f_1, f_2, \ldots, f_N \} \) and a nondegenerate \( N \) variates martingale \( \{ U_1, U_2, \ldots, U_N \} \) such that

\[
X(t) = \sum_{i=1}^{N} f_i(t)U_i(t), \tag{1.12}
\]

and that

\[
\mathcal{M}_t(X) = \mathcal{M}_t(U_1, U_2, \ldots, U_N). \tag{1.13}
\]

**Remark 1.4** A system \( \{ f_1, f_2, \ldots, f_N \} \) is called a Tchebycheff system if it satisfies the condition

\[
\det(f_i(s_j)) \neq 0, \text{ for any } s_1 < s_2 < \cdots < s_N. \tag{1.14}
\]

**Definition 1.4** The representation (1.12) is the (forward) Goursat representation. If the additional property (1.13) is satisfied, the representation (1.12) is called the (forward) proper Goursat representation.
Remark 1.5 Though the proper Goursat representation is not always canonical, it can be easily reduced to the generalized canonical representation by the procedure in Pitt's paper [22, pp209-210], and we know that \( X \) has the multiplicity \( M \leq N \).

In an analogous manner to Definition 1.3, we define a backward multiple Markov process as follows.

Definition 1.5 A Gaussian process \( X = \{X(t); t \in (t_0, t_1)\} \) is said to be backward \( N \)-ple Markov, if \( \{E[X(s_i)|\mathcal{M}^{s_0}(X)], i = 1, 2, \ldots, N\} \) is linearly independent for any \( s_0 \geq s_1 > s_2 > \cdots > s_N \) and if \( \{E[X(s_i)|\mathcal{M}^{s_0}(X)], i = 1, 2, \ldots, N + 1\} \) is linearly dependent for any \( s_0 \geq s_1 > s_2 > \cdots > s_{N+1} \).

By interchanging future and past in Theorem 1.4, the following corollary can be proved.

Corollary 1.3 A Gaussian process \( X = \{X(t); t \in (t_0, t_1)\} \) is backward \( N \)-ple Markov if and only if there exist a Tchebyscheff system \( \{f_1^*, f_2^*, \ldots, f_N^*\} \) and a nondegenerate \( N \) variates backward martingale \( (U_1^*, U_2^*, \ldots, U_N^*) \) such that

\[
X(t) = \sum_{i=1}^{N} f_i^*(t)U_i^*(t), \quad (1.15)
\]

and that

\[
\mathcal{M}^t(X) = \mathcal{M}^t(U_1^*, U_2^*, \ldots, U_N^*). \quad (1.16)
\]

Definition 1.6 The representation (1.15) is the backward Goursat representation. If the additional property (1.16) is satisfied, the representation (1.15) is called the backward proper Goursat representation.
We are now ready to discuss how to form the backward proper Goursat representation. A transformation of representation from forward to backward is a useful technique for our purpose, although the transformed representation is not always proper. The next lemma provides the construction of the proper representation.

**Lemma 1.1** Let \( X(t) = \sum_{i=1}^{N} h_i(t)V_i(t) \) be a backward Goursat representation (not necessarily proper). Put \( U^*_i(t) = E[V_i(t)\mathcal{M}^i(X)] \), then \( X(t) = \sum_{i=1}^{N} h_i(t)U^*_i(t) \) is the backward proper Goursat representation.

**Proof:** The proof is given exactly in the same manner as [22, Lemma IV.1] by interchanging past and future.

We are now in position to observe a transformation from the known forward proper Goursat representation to the backward one.

Let \( X = \{X(t); t \in (t_0, t_1)\} \) be an \( N \)-ple Markov Gaussian process having the proper Goursat representation (1.12) and let \( G(t) \) be the structure matrix of \( (U_i(t)) \), that is,

\[
G(t) = (G_{ij}(t)) = (E[U_i(t)U_j(t)]).
\]

Set

\[
\begin{cases}
  h_i(t) = \sum_{j=1}^{N} G_{ij}(t)f_j(t), & i = 1, 2, \ldots, N, \\
  V_i(u) = \sum_{j=1}^{N} G^{ij}(u)U_j(u), & \text{where } (G^{ij}(u)) = G^{-1}(u).
\end{cases}
\]  

Then \( \{h_1, h_2, \ldots, h_N\} \) is a Tchebycheff system and \( (V_i(t)) \) is a nondegenerate \( N \) variates backward martingale with respect to the natural filtering: \( \mathcal{F} = \{\sigma \{V_i(u); i = 1, 2, \ldots, N, u \geq t\}; t \in (t_0, t_1)\} \). The structure matrix of \( (V_i(t)) \) is \( G^{-1}(t) \). The representation
\[ X(t) = \sum_{i=1}^{N} h_i(t) V_i(t) \]  

(1.18)

is therefore a backward Goursat representation.

**Proposition 1.1** The $N$-ple Markov property is identical with the backward $N$-ple Markov one under the conditions (M1), (M2), (M1*), and (M2*).

**Proof:** Lemma 1.1 forms the proper backward Goursat representation from (1.18), and Corollary 1.3 implies the assertion.

**Remark 1.6** The representation (1.18) is not always canonical, as is seen in the following.

**Example.** A double Markov stationary Gaussian process with the forward canonical representation

\[ X(t) = \int_{-\infty}^{t} \left\{ 2e^{-(t-u)} - e^{-3(t-u)} \right\} dB(u), \text{ where } E dB(u)^2 = du, \]

is, by the procedure (1.17), reduced to

\[ X(t) = \int_{t}^{\infty} \left\{ 3e^{t-u} - 4e^{3(t-u)} \right\} d\bar{B}(u), \text{ where } E d\bar{B}(u)^2 = du, \]

which is independent of \( \int_{t}^{\infty} e^{-5u} d\bar{B}(u) \), for any \( t \in \mathbb{R} \) (cf. Theorem 4.2).

While due to Corollary 1.1, the backward canonical representation is

\[ X(t) = \int_{t}^{\infty} \left\{ 2e^{t-u} - e^{3(t-u)} \right\} dB^*(u), \text{ where } E dB^*(u)^2 = du. \]
Lemma 1.2 \ Let \( X = \{X(t); t \in (t_0, t_1)\} \) be an \( N \)-ple Markov Gaussian process having the proper Goursat representation (1.12) and let \( G(t) \) be the structure matrix of \((U_i(t))\). Then

\[
\text{(i) } X \text{ satisfies the condition } (M2) \text{ if } G(t) \text{ tends to the zero-matrix as } t \text{ tends to } t_0, \text{ and}
\]

\[
\text{(ii) } X \text{ satisfies the condition } (M2^*) \text{ if } G^{-1}(t) \text{ tends to the zero-matrix as } t \text{ tends to } t_1.
\]

Proof: The statement (i) is obvious. So we prove only (ii).

Because (1.18) is a backward Goursat representation,

\[
\mathcal{M}^t(X) \subset \mathcal{M}^t(V_1, V_2, \ldots, V_N).
\]

Since the structure matrix \( G^{-1}(t) \) of \((V_i(t))\) tends to the zero-matrix as \( t \) tend to \( t_1 \) from the assumption, we have \( \bigcap_{t \in (t_0, t_1)} \mathcal{M}^t(V_1, V_2, \ldots, V_N) = \{0\} \). So is \( \bigcap_{t \in (t_0, t_1)} \mathcal{M}^t(X) \). This completes the proof.

Remark 1.7 \ From Lemma 1.2, we can see that the conditions \((M2)\) and \((M2^*)\) are not equivalent, unlike the stationary case.

A multiple Markov Gaussian process is characterized in terms of its covariance function.

Theorem 1.5 \cite{9, 22} \ A covariance function \( \Gamma \) is of an \( N \)-ple Markov Gaussian process \( X = \{X(t); t \in (t_0, t_1)\} \) if and only if \( \Gamma \) has the form

\[
\Gamma(t, s) = \sum_{i=1}^{N} f_i(t)h_i(s), \quad t \geq s,
\]

where \( \{f_1, \ldots, f_N\} \) and \( \{h_1, \ldots, h_N\} \) are linearly independent systems in any interval of \((t_0, t_1)\) and, moreover, are Tchebycheff systems.
Both the forward and the backward proper Goursat representations are characterized in the following theorem. Before stating the main theorem, we give some notation.

Suppose $X = \{X(t); t \in (t_0, t_1)\}$ is an $N$-ple Markov Gaussian process having a covariance (1.19). Denote by $(\mathcal{G}, \preceq)$ the semi-ordered set of the positive definite and increasing $N \times N$ structure matrices $G(t)$ satisfying the relation

$$(h_1(t), \ldots, h_N(t)) = (f_1(t), \ldots, f_N(t)) G(t),$$

whose order is regarded as $G_1(t) \preceq G_2(t)$ if and only if $G_2(t) - G_1(t)$ is nonnegative definite for almost all $t$.

**Theorem 1.6** Let $X = \{X(t); t \in (t_0, t_1)\}$ be an $N$-ple Markov Gaussian process having a covariance (1.19). Then

(i) The forward Goursat representation (1.12) is proper if and only if $(E[U_i(t)U_j(t)])$ is the minimum element of $\mathcal{G}$, and

(ii) The backward Goursat representation

$$X(t) = \sum_{i=1}^{N} h_i(t)U_i^+(t)$$

(1.20)

is proper if and only if $(E[U_i^+(t)U_j^+(t)])^{-1}$ is the maximum element of $\mathcal{G}$.

**Proof:** The statement (i) is the direct consequence of Pitt [22, Corollary IV.2]. Note that the following assertion can be proved by interchanging past and future in the statement (i):

If we denote by $(\mathcal{K}, \succeq)$ the semi-ordered set of the positive definite and decreasing $N \times N$ structure matrices $K(t)$ satisfying the relation

$$(f_1(t), \ldots, f_N(t)) = (h_1(t), \ldots, h_N(t)) K(t),$$
then (1.20) is the backward proper Goursat representation if and only if 
\[ \left( E \left[ \sum_{j=1}^{N} K_{ij}(t) V_j(t) \right] \right) \] is the minimum element of \( \mathcal{C} \).

Therefore it suffices to show that the maximum element of \( \mathcal{G} \) is the inverse of the minimum element of \( \mathcal{C} \).

There exists a one-to-one correspondence from \( \mathcal{C} \) to \( \mathcal{G} \) for which \( K(t) \in \mathcal{C} \) corresponds to \( K^{-1}(t) \in \mathcal{G} \). Denote the backward martingale associated with \( K(t) = (K_{ij}(t)) \) by \( V(t) = (V_i(t)) \), then the martingale with \( K^{-1}(t) \) is \( K^{-1}(t)V(t) = \left( \sum_{j=1}^{N} K_{ij}(t) V_j(t) \right) \).

Since the minimum element \( K_0(t) \) of \( \mathcal{C} \) is the structure matrix of \( U^*(t) = (E[V_i(t) \mathcal{M}^t(X)]) \) from Lemma 1.1, it is sufficient to prove \( K^{-1}(t) \leq K_0^{-1}(t) \) for any \( K^{-1}(t) \in \mathcal{G} \). In fact,

\[ K_0^{-1}(t) - K^{-1}(t) = E \left[ \left( K_0^{-1}(t)U^*(t) - K^{-1}(t)V(t) \right)^2 \right] \]

is nonnegative definite, so the proof of (ii) is complete.

\[ \square \]

1.4 Multiple Markov Gaussian processes in the restricted sense

We are able to study the backward canonical representation in more detail by assuming the differentiability of a multiple Markov process.

**Definition 1.7** An \( N \)-ple Markov Gaussian process is said to be \( N \)-ple *Markov in the restricted sense* if it is \( N - 1 \) times differentiable in the mean.

It is known that the \( N \)-ple Markov Gaussian process \( X \) in the restricted sense automatically has unit multiplicity [22], so it has the canonical representation

\[ X(t) = \int_{t_0}^{t} \sum_{i=1}^{N} f_i(t)g_i(u)dB(u), \]

(1.21)
where \( f_i \in C^\infty ((t_0, t_1)) \), \( g_i \in C^\infty ((t_0, t_1)) \cap L^2 ((t_0, t]) \), for all \( t \in (t_0, t_1) \), and

\[
\sum_{i=1}^{N} f_i^{(j)}(t)g_i(t) = 0, \quad j = 0, 1, 2, \ldots, N - 2, \\
\neq 0, \quad j = N - 1,
\]

where \( f_i^{(j)} \) is the \( j \)th derivative of \( f_i \).

**Remark 1.8** The system \( \{f_1, f_2, \ldots, f_N\} \) is viewed as a fundamental system of solutions of the \( N \)th order ordinary differential operator \( L_t \)

\[
L_t f(t) = \frac{W(f_1, f_2, \ldots, f_N, f)(t)}{\sum_{i=1}^{N} f_i^{(N-1)}(t)g_i(t)W(f_1, f_2, \ldots, f_N)(t)}, \quad (1.22)
\]

where \( W \) means the Wronskian.

Since \( X \) is an \( N \)-ple Markov process, \( \{f_1, f_2, \ldots, f_N\} \) is a Tchebycheff system. Hence \( L_t \) is factorable in the sense of Karlin [15, pp.278-279], that is to say, there exist \( (N+1) \) positive functions \( w_0, w_1, \ldots, w_N \) such that

\[
L_t = \frac{1}{w_0} \frac{d}{dt} \frac{1}{w_1} \frac{d}{dt} \cdots \frac{1}{w_{N-1}} \frac{d}{dt} \frac{1}{w_N} \frac{d}{dt}. \quad (1.23)
\]

On the other hand, \( \{g_1, g_2, \ldots, g_N\} \) is the fundamental system of solutions of the formal adjoint operator \( L'_t \) of \( L_t \), and we have the expression

\[
L'_t = \frac{1}{w_N} \frac{d}{dt} \frac{1}{w_{N-1}} \cdots \frac{d}{dt} \frac{1}{w_1} \frac{d}{dt} \frac{1}{w_0}.
\]

Using the operator \( L_t \) in (1.22), we get the relation

\[
L_t X(t) = \dot{B}(t), \quad (1.24)
\]

which should be understood as

\[
d \left( \frac{1}{w_1} \frac{d}{dt} \cdots \frac{1}{w_{N-1}} \frac{d}{dt} \frac{1}{w_N} X(t) \right) = w_0(t) dB(t).
\]
For a multiple Markov Gaussian process $X$ in the restricted sense, the representation (1.18) is always canonical, and we will obtain a concrete relation between $B$ and $B^\ast$.

**Theorem 1.7** Let $X = \{X(t); t \in (t_0, t_1]\}$ be an $N$-ple Markov Gaussian process in the restricted sense having the canonical representation (1.21). Then

$$X(t) = \int_t^{t_1} \sum_{i=1}^N h_i(t)k_i(u)dB^\ast(u)$$

is the backward canonical representation, where

$$h_i(t) = \sum_{j=1}^N G_{ij}(t)f_j(t), \quad i = 1, 2, \ldots, N,$$

$$k_i(u) = -\sum_{j=1}^N G^{ij}(u)g_j(u), \text{ where } (G^{ij}(u)) = G^{-1}(u),$$

$$dB^\ast(t) = dB(t) + dt \int_{t_0}^t \sum_{i=1}^N k_i(t)g_i(u)dB(u).$$

**Proof:** We get by a simple calculation, for $j = 0, 1, 2, \ldots, N - 1$,

$$\sum_{i=1}^N f_i^{(j)}(t)g_i(t) = -\sum_{i=1}^N h_i^{(j)}(t)k_i(t),$$

Thus we have the relation

$$L^\ast X(t) = \dot{B}^\ast(t),$$

where
\[ L_t^* h(t) = -\frac{\sum_{i=1}^{N} h_i^{(N-1)}(t) k_i(t) W(h_1, h_2, \ldots, h_N)(t)}{\sum_{i=1}^{N} h_i^{(N-1)}(t) W(h_1, h_2, \ldots, h_N)(t)}, \]

in the similar way as in (1.24). This fact means that (1.25) is canonical.

Operating with \( L_t^* \) on both sides of (1.25), we obtain (1.26) by virtue of \( L_t^* f_i(t) = k_i(t), i = 1, 2, \ldots, N. \)

**Proposition 1.2** Let a stationary Gaussian process \( Y = \{Y(t); t \in \mathbb{R}\} \) be \( N \)-ple Markov in the restricted sense. Then \( L_t^* \) is identical with \( L_t' \).

**Proof:** For an \( N \)-ple Markov stationary process in the restricted sense, it is known that \( c(\lambda) \) of (1.9) equals to \( 1/P(i\lambda) \), where \( P \) is a polynomial of degree \( N \). Thus \( L_t \) is nothing but \( P(d/dt) \) and \( c^*(\lambda) = c(\bar{\lambda}) = 1/P(-i\lambda) \) as is shown in Section 1.2. Hence \( L_t^* = P(-d/dt) \). This completes the proof. \( \square \)
Chapter 2

Interpolation problems

2.1 Multiple Markov Gaussian processes

As an application of the backward canonical representation, we will obtain a concrete solution of the interpolation problem. Note that the solution of the problem for a Gaussian process $X$ is the conditional expectation of $X(t), t \in (a, b)$, which is the projection to the closed linear manifold $M_a^b(X)$ spanned by $\{X(s); s \in (a, b)\}$.

Proposition 2.1 Let $X = \{X(t); t \in (t_0, t_1)\}$ be an N-ple Markov Gaussian process having the proper Goursat representation (1.12). Then $M_a(X) + M_b(X)$ is closed; namely, $M_a^b(X) = M_a(X) \oplus M_b(X)$, where the notation $\oplus$ means a direct sum of manifolds (not necessarily orthogonal).

Proof: The projection $M^{b/b}(X)$ of $M^b(X)$ down to $M_b(X)$ is the $N$-dimensional manifold spanned by $\{U_1(b), \ldots, U_N(b)\}$ by virtue of the $N$-ple Markov property of $X$. Clearly, $M_a(X) \cap M^{b/b}(X) = \{0\}$, for the structure matrix of $(U_i(t))$ is increasing in $t$. Therefore $M_a(X) + M^{b/b}(X)$ is closed, since $M^{b/b}(X)$ is finite dimensional. On the other hand, the orthogonal com-
lement $(M_b(X))^\perp$ is independent of $M_a(X)$, and $M_b(X)$ is a closed submanifold of $M^{b/b}(X) \oplus (M_b(X))^\perp$. Thus $M_a(X) + M_b(X)$ is closed and, so that, $M_a(X) \cap M_b(X) = \{0\}$. The proof is completed. \]  

**Theorem 2.1** Suppose that an $N$-ple Markov Gaussian process $X = \{X(t); t \in (t_0, t_1)\}$ has the forward and the backward proper Goursat representations (1.12) and (1.20), respectively. Let, for any $t_0 < a < t < b < t_1$, \{$\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N$\} be a unique system of solutions of

$$
\begin{align*}
\alpha_i(t; a, b) + \sum_{j=1}^{N} K_{ij}(b) \beta_j(t; a, b) &= f_i(t) \\
\sum_{j=1}^{N} G_{ij}(a) \alpha_j(t; a, b) + \beta_i(t; a, b) &= h_i(t), \quad i = 1, 2, \ldots, N,
\end{align*}
$$

(2.1)

where $(G_{ij}(t))$ and $(K_{ij}(t))$ denote the structure matrices of $(U_i(t))$ and $(U^*_i(t))$, respectively.

Then the interpolation is uniquely expressed in the form

$$
E \left[ X(t)|M_a^b(X) \right] = \sum_{i=1}^{N} \alpha_i(t; a, b) U_i(a) + \sum_{i=1}^{N} \beta_i(t; a, b) U^*_i(b). \tag{2.2}
$$

**Proof:** Denote by $\hat{X}(t; a, b)$ the right-hand side of (2.2). It is obvious that $\hat{X}(t; a, b) \in M^b_a(X)$. Moreover we get

$$
E \left[ (X(t) - \hat{X}(t; a, b)) X(s) \right] = \sum_{i=1}^{N} f_i(t) h_i(s) - \sum_{i,j=1}^{N} \alpha_i(t; a, b) G_{ij}(s) f_j(s) - \sum_{i,j=1}^{N} \beta_i(t; a, b) K_{ij}(b) h_j(s) \\
= \sum_{i=1}^{N} \left( f_i(t) - \alpha_i(t; a, b) - \sum_{j=1}^{N} \beta_j(t; a, b) K_{ij}(b) \right) h_i(s), \quad s \in (t_0, a],
$$
and similarly,

\[ E \left[ \left( X(t) - \tilde{X}(t; a, b) \right) X(s) \right] = \sum_{i=1}^{N} \left( h_i(t) - \sum_{j=1}^{N} \alpha_j(t; a, b) G_{ij}(a) - \beta_i(t; a, b) \right) f_i(s), \quad s \in [b, t_1). \]

The system of equations (2.1) is satisfied if and only if \( X(t) - \tilde{X}(t; a, b) \) is independent of \( X(s) \) for all \( s \in (a, b)^c \), because \( \{h_i\} \) and \( \{f_i\} \) are linearly independent. Since the coefficient matrix of the system (2.1) is nonsingular, we obtain the unique solution \( \{\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N\} \) of (2.1), which satisfies (2.2).

### 2.2 Multiple Markov Gaussian processes in the restricted sense

Next, we consider the interpolation problem for a multiple Markov Gaussian process in the restricted sense. It is interesting that our answer is derived from the canonical representation, though Pitt [21] has given the answer in another way.

The following lemma, which can easily be shown, serves to prove Theorem 2.2.

**Lemma 2.1** Suppose an \( N \)-ple Markov Gaussian process \( X = \{X(t); t \in (t_0, t_1)\} \) in the restricted sense has the forward and the backward canonical representations (1.21) and (1.25), respectively. Then \( \{f_1, \ldots, f_N, h_1, \ldots, h_N\} \) is the fundamental system of solutions of the \( 2N \)th order ordinary differential operator \( L'_t L_t \), where the operator \( L_t \) is same as in (1.22).

**Theorem 2.2** Let \( X \) be as in Lemma 2.1. And let, for any \( t_0 < a < t < b < t_1 \), each \( p_j(t) \) (resp. \( q_j(t) \)), \( j = 0, 1, \ldots, N - 1 \), be a solution of
\begin{equation*}
\begin{aligned}
\left\{
\begin{array}{l}
L_i^t L_t p_j(t) = 0 \\
p_j^{(k)}(a) = \delta_{jk}, \quad p_j^{(k)}(b) = 0, \quad k = 0, 1, \ldots, N - 1,
\end{array}
\right.
\end{aligned}
\end{equation*}

(resp. \begin{equation*}
\begin{aligned}
\left\{
\begin{array}{l}
L_i^t L_t q_j(t) = 0 \\
q_j^{(k)}(a) = 0, \quad q_j^{(k)}(b) = \delta_{jk}, \quad k = 0, 1, \ldots, N - 1.
\end{array}
\right.
\end{aligned}
\end{equation*}

Then the interpolation is uniquely expressed in the form

\begin{equation}
E \left[ X(t) | \mathcal{M}_a^b(X) \right] = \sum_{j=0}^{N-1} p_j(t) X^{(j)}(a) + \sum_{j=0}^{N-1} q_j(t) X^{(j)}(b).
\end{equation}

**Proof:** We firstly remark that \( L_t \) is factorable as in (1.23), so is \( L_i^t L_t \). This fact implies that the boundary value problem

\begin{equation}
\begin{aligned}
\left\{
\begin{array}{l}
L_i^t L_t \varphi(t) = 0 \\
\varphi^{(j)}(a) = \varphi^{(j)}(b) = 0, \quad j = 0, 1, \ldots, N - 1,
\end{array}
\right.
\end{aligned}
\end{equation}

has the unique solution \( \varphi \equiv 0 \) [15, p.292]. Hence \( \{p_0, \ldots, p_{N-1}, q_0, \ldots, q_{N-1}\} \) is uniquely determined.

Denote by \( \hat{X}(t; a, b) \) the right-hand side of (2.3). It is obvious that \( \hat{X}(t; a, b) \in \mathcal{M}_a^b(X) \). Moreover we get

\begin{equation*}
\begin{aligned}
E \left[ (X(t) - \hat{X}(t; a, b)) X(s) \right] &= \sum_{i=1}^{N} f_i(t) h_i(s) - \sum_{j=0}^{N-1} p_j(t) \sum_{i=1}^{N} f_i^{(j)}(a) h_i(s) - \sum_{j=0}^{N-1} q_j(t) \sum_{i=1}^{N} f_i^{(j)}(b) h_i(s) \\
&= \sum_{i=1}^{N} \left\{ f_i(t) - \sum_{j=0}^{N-1} \left( p_j(t) f_i^{(j)}(a) + q_j(t) f_i^{(j)}(b) \right) \right\} h_i(s), \quad s \in (t_0, a].
\end{aligned}
\end{equation*}

Put \( \varphi_i(t) = f_i(t) - \sum_{j=0}^{N-1} \left( p_j(t) f_i^{(j)}(a) + q_j(t) f_i^{(j)}(b) \right), i = 1, 2, \ldots, N \). Then \( \varphi_i(t) \) is a solution of (2.4). Hence \( \varphi_i(t) \equiv 0, i = 1, 2, \ldots, N \). This means that \( X(t) - \hat{X}(t; a, b) \) is independent of \( X(s) \) for all \( s \in (t_0, a] \).
Using a similar calculation, we know that \( X(t) - \hat{X}(t; a, b) \) is also independent of \( X(s) \) for all \( s \in [b, t_1) \). Therefore the proof is completed.

### 2.3 Reciprocal Gaussian processes

Using the results in Section 2.1, we see that some multiple Markov processes are simplest for the interpolation problem, which are none less than reciprocal processes.

**Definition 2.1** A Gaussian process \( X = \{X(t); t \in (t_0, t_1)\} \) is said to be reciprocal if the condition \( E[X(t)|\mathcal{M}_a^b(X)] = E[X(t)|X(a), X(b)] \) holds for any \( a, b \in (t_0, t_1) \) and any \( t \in (a, b) \).

**Remark 2.1** Equivalently, a Gaussian process \( X = \{X(t); t \in (t_0, t_1)\} \) is reciprocal, if \( \mathcal{M}_a^b(X) \) and \( \{X(s); s \in [a, b]\} \) are conditionally independent with respect to given data \( X(a) \) and \( X(b) \) for any \( a, b \in (t_0, t_1) \). This fact says that the reciprocal property may be regarded as a one-dimensional parameter version of the Markov one for a random field. In this sense, the reciprocal property is often called quasi-Markov property.

Let the \( N \)-ple Markov Gaussian process \( X \) have the forward and the backward proper Goursat representations (1.12) and (1.20), respectively. By virtue of Theorem 2.1, there exist \( 2N \) functions \( \alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N \) such that (2.2) holds.

On the other hand,

\[
E[X(t)|X(a), X(b)] = c(t; a, b)X(a) + d(t; a, b)X(b)
= \sum_{i=1}^{N} c(t; a, b)f_i(a)U_i(a) + \sum_{i=1}^{N} d(t; a, b)h_i(b)U_i^*(b),
\]

where \( c(t; a, b) \) and \( d(t; a, b) \) are solutions of the system of equations...
\begin{equation}
\begin{pmatrix}
\Gamma(a, a) & \Gamma(b, a) \\
\Gamma(a, b) & \Gamma(b, b)
\end{pmatrix}
\begin{pmatrix}
c(t; a, b) \\
d(t; a, b)
\end{pmatrix}
= 
\begin{pmatrix}
\Gamma(t, a) \\
\Gamma(t, b)
\end{pmatrix}, \tag{2.5}
\end{equation}

The process $X$ is reciprocal if and only if they are equal. Thanks to Proposition 2.1,

\[
\begin{cases}
\alpha_i(t; a, b) = c(t; a, b)f_i(a) \\
\beta_i(t; a, b) = d(t; a, b)h_i(b), \quad i = 1, 2, \ldots, N.
\end{cases}
\]

By noting (2.1) and Theorem 1.6,

\[
\begin{cases}
f_i(a)c(t; a, b) + f_i(b)d(t; a, b) = f_i(t) \\
h_i(a)c(t; a, b) + h_i(b)d(t; a, b) = h_i(t), \quad i = 1, 2, \ldots, N.
\end{cases} \tag{2.6}
\]

Since \{f_1, \ldots, f_N\} is a Tchebyssheff system, the system of equations (2.6) in $c$ and $d$ is solvable only if $N \leq 2$ (cf. Corollary 3.1).

(i) Case of $N = 1$.

In this case, the process $X$ is (simple) Markov. And the system of equations (2.5) is identical with (2.6). Therefore we can state that a Markov Gaussian process always has the reciprocal property.

(ii) Case of $N = 2$.

In this case, the process $X$ is double Markov. In order to exist the solutions $c$ and $d$ of (2.6), $h_1$ and $h_2$ should be linear combinations of $f_1$ and $f_2$. Then the solutions $c$ and $d$ in (2.5) always satisfy the system (2.6). There exists a matrix $C = (c_{ij})$ such that $h_i(t) = \sum_{j=1}^{2} c_{ij}f_j(t), \quad i = 1, 2$. Since $C$ is nonsingular, we can find an orthogonal matrix $P = (p_{ij})$ so that

\[
P C P^{-1} = \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}
\lambda_1 & 0 \\
\lambda_3 & \lambda_2
\end{pmatrix}.
\]

Put $\bar{f}_i(t) = \sum_{j=1}^{2} p_{ij}f_j(t), \quad i = 1, 2.$
INTERPOLATION PROBLEMS

If $P^{-1}CP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, then $\Gamma(t, s) = \sum_{i=1}^{2} \lambda_i \tilde{f}_i(t) \tilde{f}_i(s)$. This means that the given process $X$ is a linear combination of two independent random variables:

$$X(t) = \sum_{i=1}^{2} \sqrt{\lambda_i} \tilde{f}_i(t) \xi_i,$$

where $\xi_1$ and $\xi_2$ are mutually independent standard Gaussian random variables.

If $P^{-1}CP = \begin{pmatrix} \lambda_1 & 0 \\ \lambda_3 & \lambda_2 \end{pmatrix}$, then $\Gamma(t, s) = \sigma(t)\sigma(s) + \tilde{\Gamma}(t, s)$, where $\sigma(t) = \sqrt{\lambda_1} \tilde{f}_1(t)$ and $\tilde{\Gamma}(t, s) = \tilde{f}_2(t)\{\lambda_3 \tilde{f}_1(s) + \lambda_2 \tilde{f}_2(s)\}$. Since it can be proved that $\tilde{\Gamma}$ is positive definite, the given process $X$ is an independent sum of a Gaussian random variable and a Markov Gaussian process:

$$X(t) = \sigma(t)\xi + \tilde{X}(t),$$

where $\xi$ is a standard Gaussian random variable, $\tilde{X}$ is a Markov Gaussian process, and they are mutually independent (cf. Corollary 3.2).

Conversely, the following theorem is known:

**Theorem 2.3** [5, 20] If a mean-continuous centered Gaussian process $X = \{X(t); t \in (t_0, t_1)\}$ is reciprocal, then

$$\tilde{X}(t) = X(t) - E[X(t)|X(a)], \quad t \in (a, b),$$

(2.7)

is Markov for any $a, b \in (t_0, t_1)$. 

Chapter 3

Multiple reciprocal property for multiple Markov Gaussian processes

3.1 Definitions

Let us define a multiple reciprocal property as a generalization of the reciprocal property defined in the previous section.

Definition 3.1 A Gaussian process \( X = \{X(t); t \in (t_0, t_1)\} \) is said to be \( L \)-ple reciprocal if there exists the least number \( L \in \mathbb{N} \) such that, for any \( s_0 < \cdots < s_L \) and any \( t \in (s_0, s_L) \), there is a random variable \( U(t; s_0, \ldots, s_L) \), which is \( \sigma\{X(s); s = s_0, \ldots, s_L\} \)-measurable, satisfying that \( X(t) - U(t; s_0, \ldots, s_L) \) is independent of \( \mathcal{M}_{s_0}^t(X) \).

Remark 3.1 The \( (2L-1) \)-ple reciprocal property is regarded as a rephrased definition of the \( L \)-ple Markov property in the sense of Kubo [16, p.65] for a one-dimensional
parameter random field.

It is noted that the class of simple reciprocal processes is identical with that of (ordinary) reciprocal processes.

The reader may think that the following definition is more natural for the generalization of the reciprocal property:

**Definition 3.2** A Gaussian process \( X = \{X(t); t \in (t_0, t_1)\} \) is said to be \( n \)-ple quasi-Markov if \( X(t) - E[X(t)|X(s); s = s_0, s_1, \ldots, s_n] \) is independent of \( \mathcal{M}_{s_0}^n(X) \) for any \( s_0 < \cdots < s_n \) and any \( t \in (s_0, s_n) \).

**Remark 3.2** A class of simple quasi-Markov processes is identical with that of (ordinary) reciprocal ones.

Indeed, we are able to characterize the multiple quasi-Markov process using terminology of the covariance function:

**Theorem 3.1** A Gaussian process \( X = \{X(t); t \in (t_0, t_1)\} \) is \( n \)-ple quasi-Markov if and only if the covariance function \( \Gamma(t, s) = E[X(t)X(s)] \) satisfies the condition

\[
\begin{vmatrix}
\Gamma(t, u) & \Gamma(s_0, u) & \cdots & \Gamma(s_n, u) \\
\Gamma(t, s_0) & \Gamma(s_0, s_0) & \cdots & \Gamma(s_n, s_0) \\
\vdots & \vdots & \ddots & \vdots \\
\Gamma(t, s_n) & \Gamma(s_0, s_n) & \cdots & \Gamma(s_n, s_n)
\end{vmatrix} = 0,
\]

(3.1)

for any \( s_0 < \cdots < s_n \), any \( t \in (s_0, s_n) \) and any \( u \in (s_0, s_n)^c \).

**Proof:** Put \( \sum_{j=0}^n a_j X(s_j) = E[X(t)|X(s); s = s_0, s_1, \ldots, s_n] \). From the definition, \( X(t) - \sum_{j=0}^n a_j X(s_j) \) is independent of \( X(s_j), j = 0, 1, \ldots, n \), and \( X(u), u \in (s_0, s_n)^c \), if \( X \) is \( n \)-ple quasi-Markov. Therefore the system of equations

\[
\begin{align*}
\Gamma(t, u) + \sum_{j=0}^n \Gamma(s_j, u) a_j &= 0, & u \in (s_0, s_n)^c \\
\Gamma(t, s_i) + \sum_{j=0}^n \Gamma(s_j, s_i) a_j &= 0, & i = 0, 1, \ldots, n,
\end{align*}
\]
holds. Since the solution \( \{1, a_0, \ldots, a_n\} \) of this system of equations is nontrivial, the coefficient matrix is singular. This means the equation (3.1) holds.

Conversely, if (3.1) holds, \( \Gamma(t, u) = a_0 \Gamma(s_0, u) + \cdots + a_n \Gamma(s_n, u) \) by expanding by the first row, for

\[
\text{det} \left( \begin{array}{cc}
\Gamma(s_0, s_0) & \Gamma(s_n, s_0) \\
\cdots & \cdots \\
\Gamma(s_0, s_n) & \Gamma(s_n, s_n)
\end{array} \right) \neq 0.
\]

This means

\[
E \left[ \left( X(t) - \sum_{j=0}^{n} a_j X(s_j) \right) X(u) \right] = E [(X(t) - E[X(t)|X(s); s = s_0, s_1, \ldots, s_n]) X(u)] = 0.
\]

Therefore the process \( X \) is \( n \)-ple quasi-Markov.

However, the author conceives that Definition 3.2 is too restrictive.

### 3.2 Multiple reciprocal Gaussian processes

In this section, we characterize \( L \)-ple reciprocal \( N \)-ple Markov processes using terminology of covariance functions.

**Theorem 3.2** Let a Gaussian process \( X = \{X(t); t \in (t_0, t_1)\} \) be \( N \)-ple Markov having the covariance function (1.19). Then \( X \) is \( L \)-ple reciprocal if and only if the relation

\[
\text{rank} \left( \begin{array}{ccc}
f_1(s_0) & \cdots & f_1(s_{L+1}) \\
\cdots & \cdots & \cdots \\
f_N(s_0) & \cdots & f_N(s_{L+1}) \\
h_1(s_0) & \cdots & h_1(s_{L+1}) \\
\cdots & \cdots & \cdots \\
h_N(s_0) & \cdots & h_N(s_{L+1})
\end{array} \right) = L + 1 \quad (3.2)
\]
holds for any distinct \((L+2)\) points.

**Proof:** From Definition 3.1, \(X\) is at most \(L\)-ple reciprocal if and only if there exists a random variable

\[
U(t; s_0, \ldots, s_L) = \sum_{j=0}^{L} a_j(t; s_0, \ldots, s_L)X(s_j)
\]

satisfying

\[
E[(X(t) - U(t; s_0, \ldots, s_L))X(s)] = 0, \text{ for any } s \in (s_0, s_L)^c,
\]

that is to say,

\[
\Gamma(t, s) - \sum_{j=0}^{L} a_j(t; s_0, \ldots, s_L)\Gamma(s_j, s) = 0, \text{ for any } s \in (s_0, s_L)^c,
\]

\[\text{ (3.3)}\]

in special,

\[
\sum_{i=1}^{N} \left\{ f_i(t) - \sum_{j=0}^{L} a_j(t; s_0, \ldots, s_L)f_i(s_j) \right\} h_i(s) = 0, \quad s \in (t_0, s_0].
\]

Since \(\{h_1, \ldots, h_N\}\) is a linearly independent system,

\[
f_i(t) - \sum_{j=0}^{L} a_j(t; s_0, \ldots, s_L)f_i(s_j) = 0, \quad i = 1, 2, \ldots, N.
\]

\[\text{ (3.4)}\]

By the same method, we obtain

\[
h_i(t) - \sum_{j=0}^{L} a_j(t; s_0, \ldots, s_L)h_i(s_j) = 0, \quad i = 1, 2, \ldots, N.
\]

\[\text{ (3.5)}\]

The system of equations (3.4) and (3.5) in \(a_0(t; s_0, \ldots, s_L), a_1(t; s_0, \ldots, s_L), \ldots, a_L(t; s_0, \ldots, s_L)\) is solvable if and only if the relation
\[
\begin{pmatrix}
  f_1(s_0) & \ldots & f_1(s_L) \\
  \vdots & \ddots & \vdots \\
  f_N(s_0) & \ldots & f_N(s_L) \\
  h_1(s_0) & \ldots & h_1(s_L) \\
  \vdots & \ddots & \vdots \\
  h_N(s_0) & \ldots & h_N(s_L)
\end{pmatrix}
= \text{rank}
\begin{pmatrix}
  f_1(s_0) & \ldots & f_1(s_L) & f_1(t) \\
  \vdots & \ddots & \vdots & \vdots \\
  f_N(s_0) & \ldots & f_N(s_L) & f_N(t) \\
  h_1(s_0) & \ldots & h_1(s_L) & h_1(t) \\
  \vdots & \ddots & \vdots & \vdots \\
  h_N(s_0) & \ldots & h_N(s_L) & h_N(t)
\end{pmatrix}
\tag{3.6}
\]

holds for any distinct \((L+2)\) points: \(s_0, \ldots, s_L, t\). Therefore, the least number \(L\) satisfying (3.6) should also satisfy (3.2).

**Corollary 3.1** Let a Gaussian process \(X\) be \(N\)-ple Markov. Then

(i) the process \(X\) is at most \((2N - 1)\)-ple reciprocal.

(ii) the process \(X\) is at least \((N - 1)\)-ple reciprocal.

**Proof:** As the system \(\{f_1, \ldots, f_N\}\) is a Tchebycheff system, the left-hand side of (3.2) is more than \(N\) and less than \(2N\). So the proof is obvious.

The evaluation above is best possible. In fact, an \(N\)-ple Markov Gaussian process in the restricted sense is \((2N - 1)\)-ple reciprocal, because the system \(\{f_1, \ldots, f_N, h_1, \ldots, h_N\}\) of a covariance function (1.19) is a Tchebycheff system as we have seen in Lemma 2.1. And we will see in the following corollary that there exists an \((N - 1)\)-ple reciprocal \(N\)-ple Markov process.

**Corollary 3.2** Let a Gaussian process \(X\) be \(N\)-ple Markov. Then \(X\) is \((N - 1)\)-ple reciprocal if and only if \(X\) has an expression

\[
X(t) = \sigma(t)\xi + \tilde{X}(t),
\tag{3.7}
\]

where \(\xi\) is a standard Gaussian random variable, \(\tilde{X}\) is an \((N - 1)\)-ple Markov Gaussian process, and they are mutually independent.
PROOF: By virtue of Theorem 3.2, \(X\) is \((N - 1)\)-ple reciprocal having the covariance function (1.19) if and only if \(h_1, \ldots, h_N\) are linear combinations of \(\{f_1, \ldots, f_N\}\), i.e., there exists a matrix \(C = (c_{ij})\) such that

\[
h_i(t) = \sum_{j=1}^{N} c_{ij} f_j(t), \quad i = 1, \ldots, N. \tag{3.8}
\]

As the matrix \(C\) is clearly nonsingular, there exists an orthogonal matrix \(P = (p_{ij})\) such that \(PCP^{-1}\) is a lower triangular matrix. Put \(\tilde{f}_i(t) = \sum_{j=1}^{N} p_{ij} f_j(t)\), \(i = 1, \ldots, N\). Then the covariance function \(\Gamma\) of \(X\) can be rewritten as

\[
\Gamma(t, s) = \sigma(t)\sigma(s) + \tilde{\Gamma}(t, s), \quad t \geq s, \tag{3.9}
\]

where \(\tilde{\Gamma}(t, s) = \sum_{i=1}^{N-1} \tilde{f}_i(t)\tilde{h}_i(s)\), so that the functions \(\tilde{h}_1, \ldots, \tilde{h}_{N-1}\) are linear combinations of \(\{\sigma, \tilde{f}_1, \ldots, \tilde{f}_{N-1}\}\). \(\Gamma\) is positive definite, so is \(\tilde{\Gamma}\). Therefore \(X\) is expressed as (3.7).

PROPOSITION 3.1 Let \(X\) be as in Theorem 3.2. Suppose there exist \(k\) functions \(\hat{h}_1, \ldots, \hat{h}_k \in \{h_1, \ldots, h_N\}\) such that

(i) \(\{f_1, \ldots, f_N, \hat{h}_1, \ldots, \hat{h}_k\}\) is a Tchebycheff system, and that

(ii) any \(h \in \{h_1, \ldots, h_N\} \setminus \{\hat{h}_1, \ldots, \hat{h}_k\}\) is a linear combination of \(\{f_1, \ldots, f_N, \hat{h}_1, \ldots, \hat{h}_k\}\).

Then \(X\) is \((N + k - 1)\)-ple reciprocal.

PROOF: From the conditions (i) and (ii), the left-hand side of (3.2) is equal to \(\min(N + k, L + 2)\). Thus \(X\) is \(L\)-ple reciprocal if and only if \(N + k = L + 1\). Therefore the proof is complete.
Chapter 4

The noncanonical representations for stationary Gaussian processes

4.1 Noncanonical representations for stationary processes

Let a stationary Gaussian process $Y = \{Y(t); t \in \mathbb{R}\}$ satisfying (M2) have a moving average representation

$$Y(t) = \int_{-\infty}^{t} F(t-u)dB(u), \quad t \in \mathbb{R}. \quad (4.1)$$

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In such a case, it is known that the spectral measure is absolutely continuous and that the spectral density $f(\lambda)$ satisfies (1.7) and is in the Hardy class $H_2$ in the lower half plane. Then the inverse Fourier transform $c(\lambda)$ of $F$ is uniquely decomposed into the form

$$c(\lambda) = Cc_0(\lambda)c_1(\lambda)c_2(\lambda)c_3(\lambda),$$

(4.2)

with notations

$$C = e^{i\gamma},$$
$$c_0(\lambda) = \sqrt{2\pi} \exp \left\{ -\frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{1 + \omega \lambda \log f(w)}{w - \lambda} \frac{d\omega}{1 + \omega^2} dw \right\},$$
$$c_1(\lambda) = \prod_{n=1}^{\infty} \frac{\lambda - z_n}{\lambda - \overline{z}_n} \frac{|z_n + i|}{|z_n - i|} \frac{|z_n - i|}{|z_n + i|},$$
$$c_2(\lambda) = e^{i\alpha\lambda},$$
$$c_3(\lambda) = \exp \left\{ \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{1 + \omega \lambda}{w - \lambda} d\beta(w) \right\},$$

where $\gamma$ is a real constant, $\{z_n\}$ is a complex sequence satisfying $\Re(z_n) > 0$ and $\sum_{n=1}^{\infty} \Re(z_n)/(1 + |z_n|^2) < \infty$, $\alpha$ is a nonnegative constant and $\beta(w)$ is a non-decreasing function of bounded variation whose derivative is zero for almost all $w$.

**Remark 4.1** Due to Beurling [1], $c_0(\lambda)$ and $c_1(\lambda)c_2(\lambda)c_3(\lambda)$ are called the *outer function* and the *inner function* for $c(\lambda)$, respectively. In addition, we call $c_1(\lambda)$, $c_2(\lambda)$ and $c_3(\lambda)$ the *Blaschke part*, the *delay part* and the *singular part*, respectively.

The inner function have unit modulus along the real axis, so that

$$|c(\lambda)| = |c_0(\lambda)|, \quad \lambda \in \mathbb{R}.$$ 

(4.3)
Also we have

$$|c_2(\lambda)| < 1, \quad \Im(\lambda) > 0, j = 1, 2, 3.$$  \hspace{1cm} (4.4)

The following theorem gives a standard for the theory of the canonical representation of a stationary Gaussian process.

**Theorem 4.1** [14, Hilfssatz 3] The representation (4.1) of a stationary Gaussian process is canonical if and only if the inner part of $c(\lambda)$ is trivial i.e. $c(\lambda) = Cc_0(\lambda)$.

This means that the noncanonical property is governed by the inner part. As we are concerned about a noncanonical representation of a Gaussian stationary process in this chapter, the inner function in (4.2) is important. Let us give some information for the inner function.

The delay part $c_2(\lambda)$ means the time shift by $\alpha$. In detail, if $\tilde{F}$ has the inverse Fourier transform $\tilde{c}(\lambda) = Cc_0(\lambda)c_1(\lambda)c_3(\lambda)$ and $\tilde{Y}$ is represented by

$$\tilde{Y}(t) = \int_{-\infty}^{t} \tilde{F}(t - u)dB(u),$$

then the representation corresponding to $c(\lambda)$ of (4.2) is given by

$$Y(t) = \tilde{Y}(t - \alpha) = \int_{-\infty}^{t-\alpha} \tilde{F}(t - \alpha - u)dB(u).$$  \hspace{1cm} (4.5)

Therefore each $B(t + h) - B(t), \ 0 < h \leq \alpha$, is orthogonal to $\mathcal{M}_t(\dot{B})$, so the term has little importance.

As for the singular part $c_3(\lambda)$, we have not yet got a clarified result on what kind of a noncanonical property the part brings about.

The Blaschke part $c_1(\lambda)$ undertakes the zero-points of $c(\lambda)$ in the upper half-plane. The following theorem makes a definite relation to the orthogonal complement of $\mathcal{M}_t(Y)$ in $\mathcal{M}_t(\dot{B})$ for each $t \in \mathbb{R}$. 
THEOREM 4.2 Suppose that a stationary Gaussian process $Y$ have a representation (4.1) and that the inverse Fourier transform $c(\lambda)$ of $F$ is given by (4.2). Then the random variable $\int_{-\infty}^{t} e^{-i\lambda_0 u} dB(u)$ in $\mathcal{M}_t(\hat{B})$ is orthogonal to $\mathcal{M}_t(Y)$ for each $t \in \mathbb{R}$, if and only if the point $\lambda_0$ is a zero-point of $c(\lambda)$ in the upper half-plane.

PROOF: The convolution of $F(u)$ and $e^{-izu}$ is nothing but $c(z)$ times $e^{-izt}$, so the statement is obvious.

Applying the theorem above, we can construct a noncanonical representation of $Y$ having a nontrivial orthogonal complement in $\mathcal{M}_t(\hat{B})$. If the inverse Fourier transform $c(\lambda)$ of $F$ is given by

$$c(\lambda) = \frac{1}{1 - i\lambda} \prod_{n=1}^{\infty} \frac{p_n + i\lambda}{p_n - i\lambda} \frac{|p_n - 1|}{p_n - 1},$$

(4.6)

where the sequence $\{p_n\}$ satisfies

$$\sum_{n=1}^{\infty} \frac{p_n}{1 + p_n^2} < \infty \text{ and } p_n > 0, \quad n \in \mathbb{N},$$

(4.7)

then

$$\mathcal{M}_t(Y) \perp \left\{ \int_{-\infty}^{t} e^{p_n u} dB(u); n \in \mathbb{N} \right\}, \quad t \in \mathbb{R},$$

(4.8)

by virtue of Theorem 4.2. The condition (4.7) guarantees the convergence of the Blaschke product in (4.6).

REMARK 4.2 The space of the right-hand side of (4.8) is, of course, smaller than the closed linear manifold (1.3), because the orthogonal complement is $\mathcal{M}_t(Y)$. This means that $\{e^{p_n u}\}$ satisfying (4.7) is not complete in $L^2((-\infty, t])$ for any $t \in \mathbb{R}$. This fact is known as Münts-Szász's theorem.
NONCANONICAL REPRESENTATIONS

Remark 4.3 The outer part $\frac{1}{1-t\lambda}$ of (4.6) is the inverse Fourier transform of $F(t) = e^{-t}$. Since

$$Y_0(t) = \int_{-\infty}^{t} e^{-(t-u)}dB(u), \quad t \in \mathbb{R},$$

is the canonical representation of the Ornstein-Uhlenbeck process, the example gives a noncanonical representation of the Ornstein-Uhlenbeck process.

4.2 Infinite dimensional orthogonal complement

In this section, we consider the noncanonical representations in the framework of the innovation theorem that gives a method for extracting the white noise.

Theorem 4.3 [12] Let $\hat{B}(t)$ be a white noise and let $\varphi(t)$ be a $\sigma\{B(s); s \leq t\}$-measurable process, for each $t \in \mathbb{R}$, satisfying $E \left[ \int_{a}^{b} |\varphi(t)| dt \right] < \infty$, for any $a < b \in \mathbb{R}$. For a process $\dot{X}(t) = \hat{B}(t) + \varphi(t),$

$$\dot{\hat{B}}(t) = \dot{X}(t) - E[\varphi(t)|\mathcal{M}_t(\dot{X})], \quad t \in \mathbb{R},$$

is a white noise again.

In this section, it is assumed that the process $\varphi = \{\varphi(t, \omega); t \in \mathbb{R}\}$ is stationary correlated with the white noise $\hat{B}$ constituting a Gaussian system $\{\hat{B}, \varphi\}$. Under the assumption, $\varphi$ is expressed in the form

$$\varphi(t) = -\int_{-\infty}^{t} l(t-u)dB(u), \quad l \in L^2(\mathbb{R}_+).$$

We will consider a process defined by

$$Y(t) \equiv \int_{-\infty}^{t} e^{-(t-u)}dX(u)$$
\[ = \int_{-\infty}^{t} e^{-(t-u)} \left\{ 1 - \int_{0}^{t-u} l(v)e^{v}dv \right\} dB(u). \quad (4.9) \]

**Remark 4.4** Since the kernel function above is clearly supported by a whole positive axis, the representation of the type of (4.9) never has the delay part.

**Proposition 4.1** If the process defined by (4.9) satisfies the orthogonal property (4.8), then \( \{ p_n \} \) tends to zero as \( n \) tends to infinity.

**Proof:** By the assumption, we have

\[ 1 = \int_{0}^{\infty} e^{-p_n x} l(x)dx. \quad (4.10) \]

Applying Schwarz's inequality, we see \( \int_{0}^{\infty} l^2(x)dx \geq 2p_n, \quad n \in \mathbb{N} \), from which it is concluded that \( \{ p_n \} \) never tends to infinity. On the other hand, \( \{ p_n \} \) enjoys (4.7). Therefore we can obtain the desired result.

On the other hand, an inner function has a following property in general.

**Proposition 4.2** [4, p.55] If the modulus of the inner function is continuous in the closed upper half-plane \( \overline{C_+} \), then the cluster point of zero-points of \( c_1(\lambda) \) is only point at infinity, if any.

**Remark 4.5** Under the assumption of Proposition 4.2, it is known that \( c_3(\lambda) \equiv 1 \) holds automatically, so we need not to pay attention to the singular part at least in the following theorem.

**Theorem 4.4** Let a process \( Y \) defined by (4.9) satisfy the orthogonal property (4.8) and let the outer function corresponding to \( Y \) have no zero-points. Then the function \( l \) in the representation (4.9) does not belong to \( L^1(\mathbb{R}_+) \).

**Proof:** Suppose \( l \) belongs to \( L^1(\mathbb{R}_+) \). Then the inverse Fourier transform of the representation kernel of \( Y \) given by (4.9) is
\[
\frac{1}{1 - i\lambda} \{1 - \overline{I}(\lambda)\},
\]
and furthermore, the inverse Fourier transform \(\overline{I}(\lambda)\) of \(I\) is continuous in the closed upper half-plane \(\overline{C}_+\). Thus the inner function of the process \(Y\) is continuous there. Therefore \(c_3 \equiv 1\) and the zero-points of \(c_1\) tend to infinity. Noting that the zero-points of an inner function accumulate either infinity or zero when they are pure imaginary, we can obtain the desired result by Proposition 4.1 and Theorem 4.2.

The theorem above tells us that the Ornstein-Uhlenbeck process \(Y\) having an infinite dimensional orthogonal complement given in page 44 cannot be of the form (4.9) with a summable function \(l\).

### 4.3 Lévy's noncanonical representations of a Brownian motion

Lévy [18] gave some examples of noncanonical representations for a Brownian motion. Let us present them here.

It is easy to prove that a Gaussian process \(\tilde{B}_q = \{\tilde{B}_q(t); t \in \mathbb{R}_+\}\) defined by

\[
\tilde{B}_q(t) = \int_0^t \left( \frac{2q + 1}{q} \frac{u^q}{t^q} - \frac{q + 1}{q} \right) dB(u),
\]

where \(B\) is a given Brownian motion, is again a Brownian motion for any \(q > -1/2\) but \(q = 0\). The representation of the right-hand side of (4.11) is not canonical, since the random variable \(\int_0^t u^q dB(u)\) is orthogonal to \(\mathcal{M}_t(\tilde{B}_q)\).

Even for \(q = 0\), we offer a noncanonical representation of a Brownian motion

\[
\tilde{B}_0(t) = \int_0^t \left( 1 + \log \frac{u}{t} \right) dB(u).
\]

For the process \(\tilde{B}_0\), \(B(t)\) itself is orthogonal to \(\mathcal{M}_t(\tilde{B}_0)\) for each \(t \in \mathbb{R}_+\).
Remark 4.6. We remark that, for any $q > -1/2$, these examples are expressed in the form

$$
\tilde{B}_q(t) = B(t) - (2q + 1) \int_0^t \int_0^s \frac{u^q}{s^{q+1}} dB(u) ds.
$$

(4.13)

Let $Y$ be a stationary Gaussian process given by (4.1). Then, by using the inverse transform of (1.11), $Y$ is transformed into $X = \{X(t); t \in \mathbb{R}_+\}$:

$$
X(t) = \sqrt{2t} Y \left( \frac{1}{2} \log t \right)
= \int_0^t \frac{1}{\sqrt{u}} \left( \frac{1}{2} \log \frac{t}{u} \right) dB_1(u),
$$

(4.14)

where $d B_1(u) = \sqrt{2u} dB \left( \frac{1}{2} \log u \right)$.

We can easily obtain the relation between the noncanonical property of the representation (4.1) and that of the representation (4.14), as follows.

Proposition 4.3. The process $Y$ of (4.1) satisfies

$$
\mathcal{M}_t(Y) \perp \int_{-\infty}^t e^{pu} dB(u), \quad t \in \mathbb{R}, \text{ for some } p > 0,
$$

if and only if the process $X$ of (4.14) satisfies

$$
\mathcal{M}_s(X) \perp \int_0^s u^q dB_1(u), \quad s \in \mathbb{R}_+, \text{ for } q = (p - 1)/2.
$$

Thanks to the proposition above and Theorem 4.2, we can construct a noncanonical representation of $X$ having the property

$$
\mathcal{M}_s(X) \perp \int_0^s u^{q_0} dB_1(u), \quad s \in \mathbb{R}_+,
$$
for any $N \in \mathbb{N}$ and any $q_n > -1/2$, $n = 1, 2, \ldots, N$, by arranging the zero-points $p_n = 2q_n + 1$ in the Blaschke part $c_1(\lambda)$ of $c(\lambda)$.

If $Y$ is of the form (4.9), then the transformed process $X$ is expressed in the form

$$X(t) = B_1(t) - \int_0^t \frac{1}{s} \int_0^s h\left(\frac{u}{s}\right) dB_1(u) ds,$$  \hspace{1cm} (4.15)

where

$$h(x) = \begin{cases} \frac{1}{2\sqrt{x}} \left(-\frac{1}{2} \log x\right), & 0 < x < 1, \\ 0, & \text{otherwise} \end{cases}$$

which belongs to $L^2((0, 1))$.

This includes Lévy's examples (4.13). Corresponding to (4.6), the noncanonical representation with infinite dimensional orthogonal complement can also be obtained. According to Theorem 4.4, the function $h(x)/\sqrt{x}$ of (4.15) does not belong to $L^1((0, 1))$, if $\mathcal{M}_t(X) \perp \left\{ \int_0^t u^{q_n} dB_1(u); n \in \mathbb{N} \right\}$. 

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