The backward canonical representations and interpolations for multiple Markov Gaussian processes

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The author gives relations between the forward canonical representation (in the sense of Lévy–Hida–Cramér) and the backward one for a Gaussian process. By using both representations, a concrete expression of the interpolation is obtained for a multiple Markov Gaussian process.

canonical representations * backward canonical representations * multiple Markov Gaussian processes * Goursat representations * interpolations

Introduction

The canonical representation theory of Gaussian processes has been originated by Lévy (1956) and systematically developed by Hida (1960) and Cramér. The idea of this representation is to express a given Gaussian process in terms of Brownian motion, the properties of which are well known, and a non-random kernel function, so that probabilistic structure of the process is represented in a visualized manner. One can therefore find good applications in the prediction problem.

The purpose of the present paper is twofold. The first is to introduce a new concept of a backward representation of Gaussian processes, for which time development is observed backward. The existing theory of representation may now be called the forward representation. Combining both backward and forward representations we are able to find profound structure of the given Gaussian process. Secondly, by applying the theory of backward representation, the interpolation problem for multiple Markov Gaussian processes is discussed. Some properties of such processes have been investigated by many authors like Mandrekar (1974) and Pitt (1975); now we go one step further to find some more interesting properties.

For these purposes, we first introduce the backward canonical representation in Section 1, having been motivated by Lévy’s work. It is convenient to observe representations of stationary processes before we come to the main topic, since the

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spectral analysis for such processes is well established in an elementary manner and ready to be used in our case.

Section 2 is devoted to multiple Markov Gaussian processes. In this case the forward canonical representation does not always have unit multiplicity, nor does the backward one. This fact causes some difficulty. The so-called Goursat representation plays an important role in our approach where we have characterized both representations. Our main result says that the backward and the forward representations are completely characterized by the use of the maximal and minimal solutions for an equation derived from the covariance function of the multiple Markov Gaussian process (Theorem 2.2).

If we assume a certain analytic property in order to have multiple Markov Gaussian processes in the restricted sense, we can prove unit multiplicity and a more explicit expression is obtained for the process in question (Section 3).

Finally, in Section 4, the interpolation of a Gaussian process will be discussed with the help of the backward canonical representation. This result is far from the general theory, however it suggests to us future questions in this direction.

1. Backward canonical representations

Consider a real valued centered Gaussian process \( X = \{X(t); \ t \in (t_0, t_1)\} \). We start with a short review of the known result related to the canonical representation for a Gaussian process according to Hida (1960). Denote by \( \mathcal{M}_t(X) \) (resp. \( \mathcal{M}^t(X) \)) the closed linear manifold spanned by \( \{X(s); \ s \leq t\} \) (resp. \( \{X(s); \ s \geq t\} \)) in the \( L^2 \)-topology. In order to exclude the discrete multiplicity, we assume the following conditions throughout the present paper:

\[
\begin{align*}
&M1: \mathcal{M}_t(X) \text{ is continuous and increasing in } t, \\
&M2: X \text{ is purely non-deterministic, that is } \bigcap_{t \in (t_0, t_1)} \mathcal{M}_t(X) = \{0\}, \\
&M1*: \mathcal{M}^t(X) \text{ is continuous and decreasing in } t, \text{ and,} \\
&M2*: \bigcap_{t \in (t_0, t_1)} \mathcal{M}^t(X) = \{0\}.
\end{align*}
\]

Remark 1.1. The conditions (M1) and (M2) do not imply (M1*) and (M2*); for example, the process \( X = \{X(t); \ t \in (0, 1)\} \) defined by

\[
X(t) = \begin{cases} B(t), & 0 < t < \frac{1}{2}, \\ B(1), & \frac{1}{2} \leq t < 1, \end{cases}
\]

where \( B \) is a Brownian motion, satisfies the conditions (M1) and (M2) but neither (M1*) nor (M2*).

Definition 1.1. (i) If, for a Gaussian process \( X \), there exist an additive Gaussian process \( B(t) \) having a spectral measure \( dv(t) = E|dB(t)|^2 \) and a function \( F(t, \cdot) \) of
$L^2((t_0, t), d\nu), t \in (t_0, t_1)$, such that $X(t)$ is expressed in the form

$$X(t) = \int_{t_0}^t F(t, u) \, dB(u),$$

(1.1)

then (1.1) is called the representation of $X$.

(ii) The representation (1.1) is called the (forward) canonical representation, if it satisfies the condition

$$\mathcal{M}_t(X) = \mathcal{M}_t(\tilde{B}), \quad t \in (t_0, t_1),$$

(1.2)

where $\mathcal{M}_t(\tilde{B})$ is the closed linear manifold spanned by \{\(B(s) - B(t_0); s \leq t\}\}.

**Remark 1.2.** (i) It is well known that

$$\mathcal{M}_t(\tilde{B}) = \left\{ \int_{t_0}^t \varphi(u) \, dB(u); \varphi \in L^2((t_0, t)) \right\}.$$

(ii) There are in general infinitely many representations of the form (1.1); among others, the canonical representation is unique; that is to say, if there exists another canonical representation

$$X(t) = \int_{t_0}^t \tilde{F}(t, u) \, dB(u), \quad \text{where } E[|dB(u)|^2] = d\tilde{\nu}(u),$$

then

$$\int_{A} F(t, u)^2 \, d\nu(u) = \int_{A} \tilde{F}(t, u)^2 \, d\tilde{\nu}(u)$$

holds for any Borel set $A \subset (t_0, t_1)$.

(iii) The property (1.2) is called proper in Hida's paper (1960).

Regarding the existence of the canonical representation, the following theorem is well known.

**Theorem 1.1** (Hida, 1960, Theorem 1.5). A Gaussian process $X = \{X(t); t \in (t_0, t_1)\}$ satisfying (M1) and (M2) has the representation

$$X(t) = \sum_{n=1}^M \int_{t_0}^t F_n(t, u) \, dB_n(u),$$

(1.3)

where

- $B_n(t)$'s, $n = 1, 2, \ldots, M \leq \infty$, are additive Gaussian processes,
- $E[|dB_m(u) \, dB_n(u)|] = \delta_{mn} \, d\nu_n(u), \quad 1 \leq m, n \leq M$,
- $d\nu_{n+1}(u) \ll d\nu_n(u)$ (absolutely continuous), $n = 1, 2, \ldots, M - 1$,
- $F_n(t, \cdot) \in L^2((t_0, t), d\nu_n)$,
with

\[ M_t(X) = \bigoplus_{n=1}^{M} M_t(B_n), \quad \text{for every } t \in (t_0, t_1). \]

\[ \square \]

**Remark 1.3.** The representation (1.3) is called the *generalized canonical representation* according to Hida (1960). It is the canonical representation if the multiplicity \( M = 1 \).

The backward canonical representation is defined analogously.

**Definition 1.2.** If, for a Gaussian process \( X \), there exist an additive Gaussian process \( B^*(t) \) having a spectral measure \( dv^*(t) = E|dB^*(t)|^2 \) and a function \( F^*(t, \cdot) \) of \( L^2((t, t_1), dv^*) \), \( t \in (t_0, t_1) \), satisfying

\[ X(t) = \int_{t}^{t_1} F^*(t, u) \, dB^*(u), \quad (1.4) \]

and the condition

\[ M'(X) = M'(\hat{B}^*), \quad t \in (t_0, t_1), \quad (1.5) \]

where \( M'(\hat{B}^*) \) is the closed linear manifold spanned by \( \{B^*(t_1) - B^*(s); s \geq t\} \), then \( X \) is said to have the *backward canonical representation* (cf. Lévy, 1956).

There exists the *generalized backward canonical representation* of a Gaussian process \( X \) under the conditions (M1*) and (M2*). This fact can be proved in analogous manner to Theorem 1.1.

The forward canonical representation theory for a stationary Gaussian process has been established and well known. With the help of this theory we can discuss the backward representation as is briefly stated below.

**Theorem 1.2** (Karhunen, 1950, Satz 5). Let \( X = \{X(t); t \in \mathbb{R}\} \) be a mean continuous stationary Gaussian process. Then the following assertions are equivalent:

(i) The condition (M2) holds.

(ii) Denote by \( f(\lambda) \) the spectral density function of \( X \),

\[ \int_{-\infty}^{\infty} \log f(\lambda) \frac{d\lambda}{1 + \lambda^2} > -\infty. \quad (1.6) \]

(iii) \( X \) has the forward canonical representation

\[ X(t) = \int_{-\infty}^{t} F(t-u) \, dB(u), \quad \text{where } E|dB(u)|^2 = du. \quad \square \quad (1.7) \]
The representation (1.7) can be constructed as follows: It is known that the function
\[
c(\lambda) = \sqrt{2\pi} \exp \left\{ -\frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{1 + w\lambda}{w - \lambda} \frac{\log f(w)}{1 + w^2} \, dw \right\}
\] (1.8)
belongs to the Hardy class \( H_2 \) in the lower half plane, and that it satisfies the relation
\[
\frac{1}{2\pi} |c(\lambda)|^2 = f(\lambda), \quad \lambda \in \mathbb{R}.
\]
And, furthermore, the Fourier transform \( F(u) \) of \( c(\lambda) \) vanishes for \( u < 0 \) by the Paley-Wiener theorem. Therefore there exists a standard Brownian motion \( B \) such that (1.7) becomes the forward canonical representation.

**Theorem 1.3.** (i) The condition (M2\( ^* \)) for a stationary process is equivalent to (M2).
(ii) Under the condition above a mean continuous stationary Gaussian process \( X = \{X(t); \ t \in \mathbb{R}\} \) has the backward canonical representation
\[
X(t) = \int_{-\infty}^{\infty} F^*(t-u) \, dB^*(u), \quad \text{where } E[|dB^*(u)|^2] = du.
\] (1.7\( ^* \))

**Proof.** (i) The condition (M2\( ^* \)) comes from the spectral property like (1.6) which is invariant under the time reflection.
(ii) Put \( c^*(\lambda) = \overline{c(\lambda)} \), then \( c^*(\lambda) \) belongs to the Hardy class in the upper half plain. The Fourier transform \( F^*(u) \) of \( c^*(\lambda) \) gives the backward canonical kernel by an analogous argument to Theorem 1.2. \( \square \)

According to Hida and Streit (1977, pp. 29-30), using the time reflection operator \( T \) defined by
\[
T1 = 1 \quad \text{and} \quad TX(t)T^{-1} = X(-t) \cdot,
\]
we can obtain the following result.

**Corollary 1.1.** For a mean continuous stationary Gaussian process \( X = \{X(t); \ t \in \mathbb{R}\} \), the forward canonical representation (1.7), and the backward one (1.7\( ^* \)) are connected in such a way that
\[
F^*(u) = F(-u),
\]
\[
T\hat{B}^*(t) = \hat{B}(t). \quad \square
\]

Applying Corollary 1.1, we obtain the backward canonical representation for a process transformable into a stationary process by a change of time scale, as follows.
**Corollary 1.2.** If a Gaussian process \( X = \{ X(t); \ t \in (0, \infty) \} \) has the forward canonical representation

\[
X(t) = \int_0^t F(t, u) \, dB(u),
\]

where the kernel \( F(\cdot, \cdot) \) is a homogeneous function of degree \( \alpha \) and \( B \) is a standard Brownian motion, then the backward canonical kernel \( F^*(\cdot, \cdot) \) is given by

\[
F^*(t, u) = F\left(t, \frac{t^2}{u}\right) \frac{t}{u}, \quad t < u.
\]

**Proof.** According to Lévy (1956), \( X \) can be transformed into a stationary process \( Y = \{ Y(t); \ t \in \mathbb{R} \} \) defined by

\[
Y(t) = e^{-2(\alpha+1)t} X(e^{2t}).
\]  

Thus the process \( Y \) has the forward canonical representation

\[
Y(t) = \int_{-\infty}^t \sqrt{2} F(1, e^{-2(t-u)}) \, e^{-(t-u)} \, dB_0(u),
\]

where

\[
\, dB_0(u) = \frac{1}{\sqrt{2}} e^u \, dB(e^{2u}).
\]

By Corollary 1.1,

\[
Y(t) = \int_t^\infty \sqrt{2} F(1, e^{2(t-u)}) \, e^{t-u} \, dB^*_0(u)
\]

is the backward canonical representation of \( Y \), which gives us the backward canonical representation of \( X \),

\[
X(t) = t^{\alpha+1/2} Y(\log \sqrt{t}) = \int_t^\infty F\left(t, \frac{t^2}{u}\right) \frac{t}{u} \, dB^*(u).
\]

\( \square \)

2. **Goursat representations for multiple Markov Gaussian processes**

Hida (1960) has defined a multiple Markov Gaussian process, as follows.

**Definition 2.1.** A Gaussian process \( X = \{ X(t); \ t \in (t_0, t_1) \} \) is called an \( N \)-ple Markov Gaussian process, if \( \{ E[X(s_i)|\mathcal{F}_s(X)], \ i = 1, 2, \ldots, N \} \) is linearly independent for any \( s_0 \leq s_1 < s_2 < \cdots < s_N \) and if \( \{ E[X(s_i)|\mathcal{F}_s(X)], \ i = 1, 2, \ldots, N+1 \} \) is linearly dependent for any \( s_0 \leq s_1 < s_2 < \cdots < s_{N+1} \).
From the definition above, a Goursat representation has been given in the following.

**Theorem 2.1** (Mandrekar, 1974; Pitt, 1975). A process $X = \{X(t); \ t \in (t_0, t_1)\}$ is an $N$-ple Markov Gaussian process if and only if there exist a Tchebycheff system \{\(f_1, f_2, \ldots, f_N\)\} and a non-degenerate $N$-ivariate martingale \((U_1, U_2, \ldots, U_N)\) such that

\[
X(t) = \sum_{i=1}^{N} f_i(t) U_i(t), \tag{2.1}
\]

and that

\[
\mathcal{M}_t(X) = \mathcal{M}_t(U_1, U_2, \ldots, U_N). \tag{2.2}
\]

**Remark 2.1.** A system \{\(f_1, f_2, \ldots, f_N\)\} is called a Tchebycheff system if it satisfies the condition

\[
\det(f_i(s_j)) \neq 0 \quad \text{for any} \ s_1 < s_2 < \cdots < s_N.
\]

**Definition 2.2.** The representation (2.1) is the (forward) Goursat representation. If the additional property (2.2) is satisfied, the representation (2.1) is called the (forward) proper Goursat representation.

**Remark 2.2.** Though the proper Goursat representation is not always canonical, it can be easily reduced to the generalized canonical representation by the procedure in Pitt (1975, pp. 209–210), and we know that $X$ has the multiplicity $M \leq N$.

In an analogous manner to Definition 2.1, we define a backward multiple Markov process as follows.

**Definition 2.3.** A Gaussian process $X = \{X(t); \ t \in (t_0, t_1)\}$ is called a backward $N$-ple Markov Gaussian process, if \(\{E[X(s_i)|\mathcal{M}_s(X)], \ i = 1, 2, \ldots, N\}\) is linearly independent for any $s_0 \geq s_1 > s_2 > \cdots > s_N$ and if \(\{E[X(s_i)|\mathcal{M}_s(X)], \ i = 1, 2, \ldots, N+1\}\) is linearly dependent for any $s_0 \geq s_1 > s_2 > \cdots > s_{N+1}$.

By interchanging future and past in Theorem 2.1, the following corollary can be proved.
Corollary 2.1. A process $X = \{X(t); t \in (t_0, t_1)\}$ is a backward $N$-ple Markov Gaussian process if and only if there exist a Tchebycheff system $\{f_1^+, f_2^+, \ldots, f_N^+\}$ and a non-degenerate $N$-variate backward martingale $(U_1^+, U_2^+, \ldots, U_N^+)$ such that
\[
X(t) = \sum_{i=1}^{N} f_i^+(t) U_i^+(t), \tag{2.3}
\]
and that
\[
\mathcal{M}'(X) = \mathcal{M}'(U_1^+, U_2^+, \ldots, U_N^+). \tag{2.4}
\]

Definition 2.4. The representation (2.3) is the backward Goursat representation. If the additional property (2.4) is satisfied, the representation (2.3) is called the backward proper Goursat representation.

We are now ready to discuss how to form the backward proper Goursat representation. A transformation of representation from forward to backward is a useful technique for our purpose, although the transformed representation is not always proper. The next lemma provides the construction of the proper representation.

Lemma 2.1. Let $X(t) = \sum_{i=1}^{N} h_i(t) V_i(t)$ be a backward Goursat representation (not necessarily proper). Put $U_i^+(t) = E[V_i(t)|\mathcal{M}'(X)]$, then $X(t) = \sum_{i=1}^{N} h_i(t) U_i^+(t)$ is the backward proper Goursat representation.

Proof. The proof is given exactly in the same manner as Lemma IV.1 in Pitt (1975) by interchanging past and future. \qed

We are now in position to observe a transformation from the known forward proper Goursat representation to the backward one.

Let $X = \{X(t); t \in (t_0, t_1)\}$ be an $N$-ple Markov Gaussian process having the proper Goursat representation (2.1) and let $G(t)$ be the structure matrix of $(U_i(t))$, that is,
\[
G(t) = (G_{ij}(t)) = (E[U_i(t) U_j(t)]).
\]

Set
\[
h_i(t) = \sum_{j=1}^{N} G_{ij}(t) f_j(t), \quad i = 1, 2, \ldots, N, \tag{2.5}
\]
\[
V_i(u) = \sum_{j=1}^{N} G^{ij}(u) U_j(u), \quad \text{where } (G^{ij}(u)) = G^{-1}(u).
\]
Then \( \{h_1, h_2, \ldots, h_N\} \) is a Tchebycheff system and \((V_i(t))\) is a non-degenerate \(N\)-variate backward martingale with respect to the natural filtering: \( \mathcal{F} = \sigma\{V_i(u); i = 1, 2, \ldots, N, u \geq t\}; \ t \in (t_0, t_1) \}. \) The structure matrix of \((V_i(t))\) is \( G^{-1}(t) \). The representation
\[
X(t) = \sum_{i=1}^{N} h_i(t) V_i(t)
\]
is therefore a backward Goursat representation.

**Proposition 2.1.** The \(N\)-ple Markov property is identical with the backward \(N\)-ple Markov one under the conditions (M1), (M2), (M1*), and (M2*).

**Proof.** Lemma 2.1 forms the proper backward Goursat representation from (2.6), and Corollary 2.1 implies the assertion. \( \square \)

**Remark 2.3.** The representation (2.6) is not always canonical, as is seen in the following.

**Example.** A double Markov stationary Gaussian process with the forward canonical representation
\[
X(t) = \int_{-\infty}^{t} \{2e^{-(t-u)} - e^{-3(t-u)}\} dB(u), \quad \text{where } E|dB(u)|^2 = du,
\]
is, by the procedure (2.5), reduced to
\[
X(t) = \int_{t}^{\infty} \{3e^{t-u} - 4e^{3(t-u)}\} d \tilde{B}(u), \quad \text{where } E|d \tilde{B}(u)|^2 = du,
\]
which is independent of \( \int_{t}^{\infty} e^{-5u} d \tilde{B}(u) \), for any \( t \in \mathbb{R} \).

While due to Corollary 1.1, the backward canonical representation is
\[
X(t) = \int_{t}^{\infty} \{2e^{t-u} - e^{3(t-u)}\} dB^*(u), \quad \text{where } E|dB^*(u)|^2 = du.
\]

**Lemma 2.2.** Let \( X = \{X(t); t \in (t_0, t_1)\} \) be an \(N\)-ple Markov Gaussian process having the proper Goursat representation (2.1) and let \( G(t) \) be the structure matrix of \((U_i(t))\). Then
(i) \( X \) satisfies the condition (M2) if \( G(t) \) tends to the zero-matrix as \( t \) tends to \( t_0 \), and
(ii) \( X \) satisfies the condition (M2*) if \( G^{-1}(t) \) tends to the zero-matrix as \( t \) tends to \( t_1 \).
**Proof.** The statement (i) is obvious. So we prove only (ii). Because (2.6) is a backward Goursat representation,

\[ \mathcal{M}'(X) \subset \mathcal{M}'(V_1, V_2, \ldots, V_N). \]

Since the structure matrix \( G^{-1}(t) \) of \( (V_i(t)) \) tends to the zero-matrix as \( t \) tend to \( t_i \), from the assumption, we have \( \bigcap_{t \in (t_0, t_i)} \mathcal{M}'(V_1, V_2, \ldots, V_N) = \{0\} \). So is \( \bigcap_{t \in (t_0, t_i)} \mathcal{M}'(X) \). This completes the proof. ☐

**Remark 2.4.** From Lemma 2.2, we can see that the conditions (M2) and (M2*) are not equivalent, unlike the stationary case.

Both the forward and the backward proper Goursat representations are characterized in the following theorem. Before stating the main theorem, we give some notation.

Suppose \( X = \{X(t); \ t \in (t_0, t_1)\} \) is an \( N \)-ple Markov Gaussian process having a covariance

\[ E[X(t)X(t')] = \sum_{i=1}^{N} f_i(t \lor t') h_i(t \land t'). \quad (2.7) \]

Denote by \( \mathcal{G} \) the semi-ordered set of the positive definite and increasing \( N \times N \) structure matrices \( G(t) \) satisfying the relation

\( (h_1(t), \ldots, h_N(t)) = (f_1(t), \ldots, f_N(t)) G(t), \)

whose order is regarded as \( G_1(t) \leq G_2(t) \) if and only if \( G_2(t) - G_1(t) \) is non-negative definite for almost all \( t \).

**Theorem 2.2.** Let \( X = \{X(t); \ t \in (t_0, t_1)\} \) be an \( N \)-ple Markov Gaussian process having a covariance (2.7). Then

(i) the forward Goursat representation (2.1) is proper if and only if \( E[U_i(t)U_j(t)] \) is the minimum element of \( \mathcal{G} \), and

(ii) the backward Goursat representation

\[ X(t) = \sum_{i=1}^{N} h_i(t) U^*_i(t) \quad (2.3') \]

is proper if and only if \( (E[U^*_i(t)U^*_j(t)])^{-1} \) is the maximum element of \( \mathcal{G} \).

**Proof.** The statement (i) is the direct consequence of Corollary IV.2 in Pitt (1975). Note that the following assertion can be proved by interchanging past and future in the statement (i):
If we denote by $\mathcal{H}$ the semi-ordered set of the positive definite and decreasing $N \times N$ structure matrices $K(t)$ satisfying the relation

$$
(f_1(t), \ldots, f_N(t)) = (h_1(t), \ldots, h_N(t))K(t),
$$

then (2.3') is the backward proper Goursat representation if and only if

$$
(E[U^*_k(t)U^*_j(t)])
$$
is the minimum element of $\mathcal{H}$.

Therefore it suffices to show that the maximum element of $\mathcal{G}$ is the inverse of the minimum element of $\mathcal{H}$.

There exists a one-to-one corresponding from $\mathcal{H}$ to $\mathcal{G}$ for which $K(t) \in \mathcal{H}$ corresponds to $K^{-1}(t) \in \mathcal{G}$. Denote the backward martingale associated with $K(t) = (K_0(t))$ by $V(t) = (V_i(t))$, then the martingale with $K^{-1}(t)$ is $K^{-1}(t)V(t) = (\sum_{j=1}^N K_j(t)V(t))$.

Since the minimum element $K_0(t)$ of $\mathcal{H}$ is the structure matrix of $U^*(t) = (E[V_i(t)M'(X)])$ from Lemma 2.1, it is sufficient to prove $K^{-1}(t) \leq K_0^{-1}(t)$ for any $K^{-1}(t) \in \mathcal{G}$. In fact,

$$
K_0^{-1}(t) - K^{-1}(t) = E[K_0^{-1}(t)U^*(t) - K^{-1}(t)V(t)]^2
$$
is non-negative definite, so the proof of (ii) is complete. □

3. Multiple Markov Gaussian processes in the restricted sense

We are able to study the backward canonical representation in more detail by assuming the differentiability of a multiple Markov process.

**Definition 3.1.** An $N$-ple Markov Gaussian process is called an $N$-ple Markov Gaussian process in the restricted sense if it is $N-1$ times differentiable in the mean.

It is known that the $N$-ple Markov Gaussian process $X$ in the restricted sense automatically has unit multiplicity (Pitt, 1975), so it has the canonical representation

$$
X(t) = \int_{t_0}^t \sum_{i=1}^N f_i(t)g_i(u) \, dB(u),
$$

where $f_i \in C^\infty((t_0, t_1))$, $g_i \in C^\infty((t_0, t_1)) \cap L^2((t_0, t_1); \text{ for all } t \in (t_0, t_1))$, and

$$
\sum_{i=1}^N f_i^{(j)}(t)g_i(t)  = 0, \quad j = 0, 1, 2, \ldots, N-2,\quad \sum_{i=1}^N f_i^{(j)}(t)g_i(t) \neq 0, \quad j = N-1,
$$

where $f_i^{(j)}$ is the $j$th derivative of $f_i$.

**Remark 3.1.** The system $\{f_1, f_2, \ldots, f_N\}$ is viewed as a fundamental system of solutions of the $N$th order ordinary differential operator $L$:

$$
L_n f(t) = \frac{W(f_1, f_2, \ldots, f_N)(f(t))}{\sum_{i=1}^N f_i^{(N-1)}(t)g_i(t) W(f_1, f_2, \ldots, f_N)(f(t))},
$$

where $W$ means the Wronskian.
Since $X$ is an $N$-ple Markov process, $\{f_1, f_2, \ldots, f_N\}$ is a Tchebycheff system. Hence $L_i$ is factorable in the sense of Karlin (1968, pp. 278–279), that is to say, there exist $N + 1$ positive functions $w_0, w_1, \ldots, w_N$ such that

$$L_i = \frac{1}{w_0} \frac{d}{dt} \frac{1}{w_1} \frac{d}{dt} \ldots \frac{1}{w_{N-1}} \frac{d}{dt} \frac{1}{w_N}.$$ 

On the other hand, $\{g_1, g_2, \ldots, g_N\}$ is the fundamental system of solutions of the formal adjoint operator $L'_i$ of $L_i$, and we have the expression

$$L'_i = \frac{1}{w_N} \frac{d}{dt} \frac{1}{w_{N-1}} \frac{d}{dt} \ldots \frac{1}{w_1} \frac{d}{dt} \frac{1}{w_0}.$$ 

Using the operator $L_i$ in (3.2), we get the relation

$$L_i X(t) = \dot{B}(t),$$

which should be understood as

$$d\left( \frac{1}{w_1} \frac{d}{dt} \ldots \frac{1}{w_{N-1}} \frac{d}{dt} \frac{1}{w_N} X(t) \right) = w_0(t) dB(t).$$

For a multiple Markov Gaussian process $X$ in the restricted sense, the representation (2.6) is always canonical, and we will obtain a concrete relation between $B$ and $B^*$.

**Theorem 3.1.** Let $X = \{X(t); t \in (t_0, t_1)\}$ be an $N$-ple Markov Gaussian process in the restricted sense having the canonical representation (3.1). Then

$$X(t) = \int_{t_0}^{t_1} \sum_{i=1}^{N} h_i(t) k_i(u) dB^*(u)$$

(3.4)

is the backward canonical representation, where

$$h_i(t) = \sum_{j=1}^{N} G_{ij}(t)f_j(t), \quad i = 1, 2, \ldots, N,$$

$$k_i(u) = -\sum_{j=1}^{N} G_{ij}(u)g_j(u), \text{ where } (G_{ij}(u)) = G^{-1}(u),$$

and

$$dB^*(t) = dB(t) + dt \int_{t_0}^{t} \sum_{i=1}^{N} k_i(t)g_i(u) dB(u).$$

(3.5)

**Proof.** We get by a simple calculation,

$$\sum_{i=1}^{N} f^{(j)}(t)g_i(t) = -\sum_{i=1}^{N} h^{(j)}(t)k_i(t) \quad \text{for } j = 0, 1, 2, \ldots, N - 1.$$
Thus we have the relation
\[ L^*_i X(t) = \dot{B}^*_i(t), \]
where
\[ L^*_i h(t) = -\frac{W(h_1, h_2, \ldots, h_N, h(t))}{\sum_{i=1}^{N} h_i^{(N-1)}(t) k_i(t) W(h_1, h_2, \ldots, h_N)(t)} \]
in the similar way as in (3.3). This fact means that (3.4) is canonical.

Operating with \( L^*_i \) on both sides of (3.4), we obtain (3.5) by virtue of \( L^*_i f_i(t) = k_i(t) \), \( i = 1, 2, \ldots, N \). \( \square \)

**Proposition 3.1.** Let a stationary Gaussian process \( X = \{X(t); \ t \in \mathbb{R}\} \) be \( N \)-ple Markov in the restricted sense. Then \( L^*_i \) is identical with \( L_i \).

**Proof.** For an \( N \)-ple Markov stationary process in the restricted sense, it is known that \( c(\lambda) \) of (1.8) equals to \( 1/P(i\lambda) \), where \( P \) is polynomial of degree \( N \). Thus \( L_i \) is nothing but \( P(d/dt) \) and \( c^*(\lambda) = c(\lambda) = 1/P(-i\lambda) \) as is shown in Section 1. Hence \( L^*_i = P(-d/dt) \). This completes the proof. \( \square \)

4. Application to interpolation problems

As an application of the backward canonical representation, we will obtain a concrete solution of the interpolation problem. Note that the solution of the problem for a Gaussian process \( X \) is the conditional expectation of \( X(t), \ t \in (a, b) \), which is the projection to the closed linear manifold \( \mathcal{M}^b(X) \) spanned by \( \{X(s); \ s \in (a, b)^2\} \).

**Proposition 4.1.** Let \( X = \{X(t); \ t \in (t_0, t_1)\} \) be an \( N \)-ple Markov Gaussian process having the proper Goursat representation (2.1). Then \( \mathcal{M}_a(X) + \mathcal{M}^b(X) \) is closed; namely, \( \mathcal{M}^a_a(X) = \mathcal{M}_a(X) \oplus \mathcal{M}^b(X) \), where the notation \( \oplus \) means a direct sum of manifolds (not necessarily orthogonal).

**Proof.** The projection \( \mathcal{M}^{b/b}(X) \) of \( \mathcal{M}^b(X) \) down to \( \mathcal{M}_b(X) \) is the \( N \)-dimensional manifold spanned by \( \{U_1(b), \ldots, U_N(b)\} \) by virtue of the \( N \)-ple Markov property of \( X \). Clearly, \( \mathcal{M}_a(X) \cap \mathcal{M}^{b/b}(X) = \{0\} \), for the structure matrix of \( (U_i(t)) \) is increasing in \( t \). Therefore \( \mathcal{M}_a(X) + \mathcal{M}^{b/b}(X) \) is closed, since \( \mathcal{M}^{b/b}(X) \) is finite dimensional. On the other hand, the orthogonal complement \( (\mathcal{M}_b(X))^\perp \) is independent of \( \mathcal{M}_a(X) \), and \( \mathcal{M}^b(X) \) is a closed submanifold of \( \mathcal{M}^{b/b}(X) \oplus (\mathcal{M}_b(X))^\perp \). Thus \( \mathcal{M}_a(X) + \mathcal{M}^b(X) \) is closed and, so that, \( \mathcal{M}_a(X) \cap \mathcal{M}^b(X) = \{0\} \). The proof is completed. \( \square \)
Theorem 4.1. Suppose an N-ple Markov Gaussian process \( X = \{X(t) ; t \in (t_0, t_1) \} \) has the forward and the backward proper Goursat representations (2.1) and (2.3'), respectively. Let, for any \( t_0 < a < t < b < t_1, \{\alpha_1, \ldots, \alpha_N ; \beta_1, \ldots, \beta_N \} \) be a unique system of solutions of

\[
\alpha_i(t) + \sum_{j=1}^{N} K_{ij}(b) \beta_j(t) = f_i(t), \\
\sum_{j=1}^{N} G_{ij}(a) \alpha_j(t) + \beta_i(t) = h_i(t), \quad i = 1, 2, \ldots, N,
\]

where \( (G_{ij}(t)) \) and \( (K_{ij}(t)) \) denote the structure matrices of \( (U_i(t)) \) and \( (U_i^*(t)) \), respectively.

Then the interpolation is uniquely expressed in the form

\[
E[X(t)|X(s); s \in (a, b)^c] = \sum_{i=1}^{N} \alpha_i(t) U_i(a) + \sum_{i=1}^{N} \beta_i(t) U_i^*(b).
\]

Proof. Denote by \( \hat{X}(t; a, b) \) the right-hand side of (4.2). It is obvious that \( \hat{X}(t; a, b) \in M_a(X) \oplus M_b(X) \). Moreover we get

\[
E[(X(t) - \hat{X}(t; a, b))X(s)]
\]

\[
= \sum_{i=1}^{N} f_i(t) h_i(s) - \sum_{i,j=1}^{N} \alpha_i(t) G_{ij}(s) f_j(s) - \sum_{i,j=1}^{N} \beta_j(t) K_{ij}(b) h_j(s)
\]

\[
= \sum_{i=1}^{N} \left( f_i(t) - \alpha_i(t) - \sum_{j=1}^{N} \beta_j(t) K_{ij}(b) \right) h_i(s), \quad s \in (t_0, a),
\]

and similarly,

\[
E[(X(t) - \hat{X}(t; a, b))X(s)]
\]

\[
= \sum_{i=1}^{N} \left( h_i(t) - \sum_{j=1}^{N} \alpha_j(t) G_{ij}(a) - \beta_i(t) \right) f_i(s), \quad s \in [b, t_1).
\]

The system of equations (4.1) is satisfied if and only if \( X(t) - \hat{X}(t; a, b) \) is independent of \( X(s) \) for all \( s \in (a, b)^c \), because \( \{h_i\} \) and \( \{f_i\} \) are linearly independent. Since the coefficient matrix of the system (4.1) is non-singular, we obtain the unique solution \( \{\alpha_1, \ldots, \alpha_N ; \beta_1, \ldots, \beta_N \} \) of (4.1), which satisfies (4.2). \( \Box \)

Finally, we consider the interpolation problem for a multiple Markov Gaussian process in the restricted sense. It is interesting that our answer is derived from the canonical representation, though Pitt (1971) has given the answer in another way.

The following lemma, which can easily be shown, serves to prove Theorem 4.2.
Lemma 4.1. Suppose an N-ple Markov Gaussian process \( X = \{X(t); t \in (t_0, t_1)\} \) in the restricted sense has the forward and the backward canonical representations (3.1) and (3.4), respectively. Then \( \{f_1, \ldots, f_N; h_1, \ldots, h_N\} \) is the fundamental system of solutions of the 2Nth order ordinary differential operator \( L_t L_s \), where the operator \( L_t \) is same as in (3.2). \( \square \)

Theorem 4.2. Let \( X \) be as in Lemma 4.1. And let, for any \( t_0 < a < t < b < t_1 \), each \( p_j(t) \) (resp. \( q_j(t) \)), \( j = 0, 1, \ldots, N - 1 \), be a solution of
\[
\begin{align*}
L_t L_s p_j(t) &= 0, \\
p_j^{(k)}(a) &= \delta_{jk}, \quad p_j^{(k)}(b) = 0, \quad k = 0, 1, \ldots, N - 1
\end{align*}
\]
(resp.
\[
\begin{align*}
L_t L_s q_j(t) &= 0, \\
q_j^{(k)}(a) &= 0, \quad q_j^{(k)}(b) = \delta_{jk}, \quad k = 0, 1, \ldots, N - 1.
\end{align*}
\]
Then the interpolation is uniquely expressed in the form
\[
E[X(t)|X(s); s \in (a, b)] = \sum_{j=0}^{N-1} p_j(t)X^{(j)}(a) + \sum_{j=0}^{N-1} q_j(t)X^{(j)}(b). \tag{4.3}
\]

Proof. Firstly we remark that \( L_t \) is factorable as in Remark 3.1, so is \( L_s L_t \). This fact implies that the boundary value problem
\[
\begin{align*}
L_t L_s \varphi(t) &= 0, \\
\varphi^{(j)}(a) &= \varphi^{(j)}(b) = 0, \quad j = 0, 1, \ldots, N - 1,
\end{align*} \tag{4.4}
\]
has the unique solution \( \varphi \equiv 0 \) (Karlin, 1968, p. 292). Hence \( \{p_0, \ldots, p_{N-1}; q_0, \ldots, q_{N-1}\} \) is uniquely determined.

Denote by \( \hat{X}(t; a, b) \) the right-hand side of (4.3). It is obvious that \( \hat{X}(t; a, b) \in M_a(X) \oplus M_b(X) \). Moreover we get
\[
E[(X(t) - \hat{X}(t; a, b))X(s)]
\]
\[
= \sum_{i=1}^{N} f_i(t)h_i(s) - \sum_{i=1}^{N} p_i(t) \sum_{j=1}^{N} f_i^{(j)}(a)h_i(s) - \sum_{j=0}^{N-1} q_j(t) \sum_{i=1}^{N} f_i^{(j)}(b)h_i(s)
\]
\[
= \sum_{i=1}^{N} \left\{ f_i(t) - \sum_{j=0}^{N-1} (p_i(t)f_i^{(j)}(a) + q_i(t)f_i^{(j)}(b)) \right\} h_i(s), \quad s \in (t_0, a].
\]
Put \( \varphi_i(t) = f_i(t) - \sum_{j=0}^{N-1} (p_i(t)f_i^{(j)}(a) + q_i(t)f_i^{(j)}(b)) \), \( i = 1, 2, \ldots, N \). Then \( \varphi_i(t) \) is a solution of (4.4). Hence \( \varphi_i = 0, i = 1, 2, \ldots, N \). This means that \( X(t) - \hat{X}(t; a, b) \) is independent of \( X(s) \) for all \( s \in (t_0, a] \).

Using a similar calculation, we know that \( X(t) - \hat{X}(t; a, b) \) is also independent of \( X(s) \) for all \( s \in [b, t_1) \). Therefore the proof is completed. \( \square \)
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References