Construction of noncanonical representations for a Gaussian process.

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1 Introduction

For a Brownian motion $B = \{ B(t); t \in \mathbb{R}_+ \}$, a centered Gaussian process $X = \{ X(t); t \in \mathbb{R}_+ \}$ is represented as

$$X(t) = \int_0^t F(t, u) dB(u), \quad t \in \mathbb{R}_+, \quad (1)$$

satisfying $H_t(X) \subseteq H_t(B)$, for any $t \in \mathbb{R}_+$, then $X$ is called canonically represented with respect to $B$. This concept was presented by Lévy in 1956, and systematically developed by Hida [2].

In this paper, we will treat those noncanonical representations of a Brownian motion itself which were presented by Lévy [5]. He gave those Brownian motions represented as (1) which satisfy $H_t(X) \notin H_t(B)$. By easy calculation, it is shown that

$$\bar{B}_q(t) = \int_0^t \left( \frac{2q + 1}{q} \frac{u^q}{t^q} - \frac{q + 1}{q} \right) dB(u), \quad 0 \neq q > -\frac{1}{2}, \quad (2)$$

is a Brownian motion satisfying $H_t(\bar{B}_q) = H_t(B) \ominus LS \left\{ \int_0^t u^q dB(u) \right\}$, for any $t \in \mathbb{R}_+$, and $\bar{B}_q$ is noncanonical with respect to $B$. Even in case of $q = 0$, we can see

$$\bar{B}_0(t) = \int_0^t \left( 1 + \log \frac{u}{t} \right) dB(u) \quad (3)$$

$$= B(t) - \int_0^t \frac{B(s)}{s} ds \quad (4)$$

is a Brownian motion satisfying $H_t(\bar{B}_0) = H_t(B) \ominus LS \{ B(t) \}$, for any $t \in \mathbb{R}_+$, and $\bar{B}_0$ is noncanonical with respect to $B$. Consequently, by rewriting (2), we can prove, for any $q > -\frac{1}{2}$ (including $q = 0$),

$$\bar{B}_q(t) = B(t) - (2q + 1) \int_0^t \int_0^s \frac{u^q}{s^{q+1}} dB(u) ds \quad (5)$$

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is a Brownian motion satisfying $H_t(\bar{B}_q) = H_t(B) \ominus LS \left\{ \int_0^t u^q dB(u) \right\}$ for any $t \in \mathbb{R}_+$. By the use of Volterra integral operator $K_q$ whose integral kernel is
\[
k_q(s, u) = \begin{cases} (2q + 1)u^q/s^{q+1}, & s \geq u, \\ 0, & s < u, \end{cases}
\]
we can represent the Brownian motion as
\[\dot{\bar{B}}_q = (I - K_q)\dot{B}, \quad \bar{B}_q(0) = 0.\] (7)

In the collaboration [1], this fact is generalized as follows: for any $g \in L^2(0, 1)$, by the use of Volterra integral operator $K_g$ whose integral kernel is
\[
k_g(s, u) = \begin{cases} g(s)g(u)/\int_0^s g(v)^2 dv, & s \geq u, \\ 0, & s < u, \end{cases}
\]
we can construct a noncanonical representation of a Brownian motion $\bar{B}_g$ with respect to $B$ such as
\[\dot{\bar{B}}_g = (I - K_g)\dot{B}, \quad \bar{B}_g(0) = 0,\] (9)
satisfying
\[H_t(\bar{B}_g) = H_t(B) \ominus LS \left\{ \int_0^t g(u)dB(u) \right\}\] (10)
for any $t \in (0, 1)$.

In Section 2, we shall treat a perturbation of the operator $K_g$. It will be proved that the representation $X_\alpha$ defined as (14) becomes noncanonical iff $\alpha > 1/2$, being independent of the choice of $g$.

In Section 3, by iterating the operator we shall construct a noncanonical representation which has two dimensional orthogonal complement. Then we consider the relation with the noncanonical representation introduced in the paper [1].

2 Perturbation of the operator

Lemma 1 Let $g \in L^2(0, 1)$.
\[g(t) \left( \int_0^t g(u)^2 du \right)^\beta \in L^2(0, 1)\] (11)
holds if and only if $\beta > -1/2$. 

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Proof: Because
\[
\int_0^1 \left\{ g(t) \left( \int_0^t g(u)^2 du \right)^\beta \right\}^2 dt = \int_0^\|g\|^2 x^{2\beta} dx, \quad (x = \int_0^t g(u)^2 du),
\]
(13)
g(t) \left( \int_0^t g(u)^2 du \right)^\beta \in L^2(0, 1) if and only if 2\beta > -1.
\]

Using the lemma above, we can see the canonicality by perturbation.

**Theorem 1** Suppose \( X_\alpha \) be defined as
\[
\dot{X}_\alpha = (I - \alpha K_g) \dot{B}, \quad X_\alpha(0) = 0,
\]
(14)
by the use of \( K_g \) in (9). Then \( X_\alpha \) is noncanonical with respect to \( B \), if and only if \( \alpha > 1/2 \).

**Proof:** Solve the equation
\[
0 = (I - \alpha K_g) \varphi(t).
\]
(15)
By differentiation
\[
\frac{\varphi(t) \int_0^t g(u)^2 du}{g(t)} = \alpha \int_0^t g(u) \varphi(u) du,
\]
(16)
\[
\frac{\varphi'(t)}{\varphi(t)} = \frac{g'(t)}{g(t)} + \frac{(\alpha - 1) g(t)^2}{\int_0^t g(u)^2 du}.
\]
(17)
By integration,
\[
\varphi(t) = C g(t) \left( \int_0^t g(u)^2 du \right)^{\alpha - 1}
\]
(18)
where \( C \) is constant. By virtue of the criterion for the canonical representation [2], this is noncanonical iff \( \varphi \in L^2 \). Because of Lemma 1, \( \varphi \in L^2 \) iff \( \alpha > 1/2 \), that is noncanonical.

\( X_\alpha \), for \( \alpha > 1/2 \), which is noncanonical with respect to \( B \), has the following canonical representation.

**Proposition 1** For \( \alpha > 1/2 \), \( X_\alpha \) in (14) is canonically represented with respect to \( B_0 \) as follows:
\[
\dot{X}_\alpha = (I - (1 - \alpha) K_g) \dot{B}_0, \quad X_\alpha(0) = 0.
\]
(19)
Proof: Since \((I - K^*_g)\) is isometry, \((I - K_g)(I - K^*_g) = I\) holds. By an easy calculation, we find
\[
\left\langle (I - \alpha K^*_g) \phi, (I - \alpha K^*_g) \psi \right\rangle = \left\langle (I - (1 - \alpha)K^*_g) \phi, (I - (1 - \alpha)K^*_g) \psi \right\rangle. \tag{20}
\]
This means that the law of \(X_{1 - \alpha}\) is same as the one of \(X_\alpha\). Thanks to Theorem 1, (19) is the canonical representation of \(X_\alpha\), because \(1 - \alpha < 1/2\). □

3 Iteration of the operator

In the article [1], for any independent functions \(g, h \in L^2(0, 1)\), we construct, a non-canonical representation of a Brownian motion \(\tilde{B}_{\{g,h\}}\) with respect to \(B\), such as
\[
\dot{\tilde{B}}_{\{g,h\}} = (I - K_{\{g,h\}})\dot{B}, \quad \tilde{B}_{\{g,h\}}(0) = 0, \tag{21}
\]
by the use of Volterra integral operator \(K_{\{g,h\}}\) whose integral kernel is
\[
k_{\{g,h\}}(s, u) = \begin{cases} 
( g(s) \quad h(s) ) \left( \begin{array}{cc}
\int_0^s g(v)^2 dv & \int_0^s g(v)h(v) dv \\
\int_0^s g(v)h(v) dv & \int_0^s h(v)^2 dv
\end{array} \right)^{-1} ( g(u) \quad h(u) ), & s \geq u, \\
0, & s < u,
\end{cases} \tag{22}
\]
having two dimensional orthogonal complement
\[
H_t(\tilde{B}_{\{g,h\}}) = H_t(B) \ominus LS \left\{ \int_0^t g(u) dB(u), \int_0^t h(u) dB(u) \right\}, \tag{23}
\]
for any \(t \in (0, 1)\).

Here, we shall construct, by different method, a noncanonical representation which has two dimensional orthogonal complement.

Theorem 2 For \(g, h \in L^2(0, 1)\), the Gaussian process \(\tilde{B}_{g,h}\) defined as
\[
\dot{\tilde{B}}_{g,h} = (I - K_{g})(I - K_{h})\dot{B}, \quad \tilde{B}_{g,h}(0) = 0, \tag{24}
\]
is a noncanonical representation of a Brownian motion with respect to \(B\), satisfying
\[
H_t(\tilde{B}_{g,h}) = H_t(B) \ominus LS \left\{ \int_0^t g(u) dB(u), \int_0^t h(u) dB(u) - \int_0^t K^*_g h(u) dB(u) \right\}, \tag{25}
\]
for any \(t \in (0, 1)\).
Proof: Put
\[
\hat{B}_g = (I - K_g)\hat{B}, \quad \bar{B}_g(0) = 0, \quad (26)
\]
then
\[
H_t(\bar{B}_g) = H_t(B) \ominus LS \left\{ \int_0^t g(u)dB(u) \right\}. \quad (27)
\]
Since \( \hat{B}_{g,h} = (I - K_h)\hat{B}_g \),
\[
H_t(\bar{B}_{g,h}) \perp LS \left\{ \int_0^t g(u)dB(u) \right\} \quad \text{in} \quad H_t(B). \quad (28)
\]
Meanwhile,
\[
H_t(\bar{B}_{g,h}) = H_t(\bar{B}_g) \ominus LS \left\{ \int_0^t h(u)d\bar{B}_g(u) \right\}. \quad (29)
\]
Since
\[
\int_0^t h(u)d\bar{B}_g(u) = \int_0^t (I - K^*_g)h(u)dB(u), \quad (30)
\]
we have proved
\[
H_t(\bar{B}_{g,h}) = H_t(B) \ominus LS \left\{ \int_0^t g(u)dB(u), \int_0^t (I - K^*_g)h(u)dB(u) \right\}. \quad (31)
\]

Especially, in case of \( g = h \), it turns iteration of the same operator.

Corollary 1 For \( g \in L^2(0,1) \), the Gaussian process \( \bar{B}_{g,g} \) defined as
\[
\hat{B}_{g,g} = (I - K_g)^2\hat{B}, \quad \bar{B}_{g,g}(0) = 0, \quad (32)
\]
is a noncanonical representation of a Brownian motion with respect to \( B \), satisfying
\[
H_t(\bar{B}_{g,g}) = H_t(B) \ominus LS \left\{ \int_0^t g(u)dB(u), \int_0^t g(u)\log \left( \int_0^u g(s)^2ds \right)dB(u) \right\} \quad (33)
\]
for any \( t \in (0,1) \).

Proof: It is obvious from
\[
K^*_g g(u) = g(u)\log g(u)^2 - g(u)\log \left( \int_0^u g(s)^2ds \right). \quad (34)
\]

\( ^\dagger \) \( g(u) \) is linearly independent of \( g(u)\log \left( \int_0^u g(s)^2ds \right) \).
Finally, we shall study a condition that \( \bar{B}_{(g,h)} \) in (21) and \( \bar{B}_{g,h} \) in (24) are identical. This means \( (I - K_h)(I - K_g) = I - K_{(g,h)} \), which corresponds to the condition of these operators to be commutative.

Thanks to Theorem 2, it is sufficient to prove that

\[
LS \left\{ \int_0^t g(u)dB(u), \int_0^t h(u)dB(u) \right\}
= LS \left\{ \int_0^t g(u)dB(u), \int_0^t h(u)dB(u) - \int_0^t K^*_g h(u)dB(u) \right\},
\]

that is to say,

\[
LS \{g, h\} = LS \{g, h - K^*_g h\}.
\]

By putting

\[
h - K^*_g h = C_1 g + C_2 h, \quad C_2 \neq 0,
\]

we get the integral equation

\[
K^*_g h = C_3 g + C_4 h, \quad C_4 \neq 1.
\]

Thus we obtain

\[
h(t) = C g(t) \left( \int_0^t g(u)^2du \right)^{C_5}, \quad C_5 \neq -1, 0,
\]

where \( C \) is constant. Therefore the proposition below is obtained.

**Proposition 2** For given \( g \in L^2(0, 1) \), the operator \( K_h \), \( h \in L^2(0, 1) \), is commutative with \( K_g \), if and only if

\[
h(t) \in LS \left\{ g(t) \left( \int_0^t g(u)^2du \right)^\gamma, \quad -\frac{1}{2} < \gamma \right\}.
\]

**Proof:** By the former discussion, the desired \( h \) is in the form (39). And the case of \( C_5 = 0 \) is clear. Thanks to Lemma 1, the statement is proved since \( h \in L^2(0, 1) \). \[\blacksquare\]
References


