Information quantity of a noncanonical representation of Gaussian processes.

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Suppose a Gaussian process \( X = \{X(t); t \in [0, \infty)\} \) to be represented as a Wiener integral

\[
X(t) = \int_0^t F(t, u)dB(u), \quad t \in [0, \infty),
\]

with a given Brownian motion \( B \). Then the relation \( H_t(X) \subseteq H_t(B) \) holds for \( t \in (0, \infty) \), where \( H_t(X) \) is the linear span of \( \{X(s); s \leq t\} \). If the equation \( H_t(X) = H_t(B) \) holds for any \( t \in [0, \infty) \), the representation (1) is called a canonical representation. The theory of canonical representation was originated by Lévy [6] and developed by Hida [3] and Crâmer [1].

If

\[
\tilde{X}(t) = \int_0^t \tilde{F}(t, u)dB(u), \quad t \in [0, \infty),
\]

is a noncanonical representation, the relation \( H_t(\tilde{X}) \not\subseteq H_t(B) \) holds. From the viewpoint of information theory, this fact is often expressed as \( \tilde{X} \) having less information than \( B \).

Meanwhile in the collaboration [2, Proposition 2.1], the author has constructed a noncanonical representation \( \tilde{X} \) of a given Gaussian process \( X \) having the canonical representation (1) such that, for any natural number \( N \) and for any linearly independent system \( g = \{g_1, g_2, \ldots, g_N\} \) in \( L^2[0, t] \),

\[
H_t(\tilde{X}) = H_t(B) \oplus LS\left\{ \int_0^t g_i(u)dB(u); i = 1, 2, \ldots, N \right\},
\]

where \( LS\{\cdots\} \) denotes the linear span of \( \{\cdots\} \). The noncanonical kernel \( \tilde{F} \) of \( \tilde{X} \) is given by

\[
\tilde{F}(t, u) = F(t, u) - \int_u^t F(t, s) \sum_{i=1}^N f_i(s) g_i(u)ds,
\]

where \( f_i(s) = \sum_{j=1}^N G^{ij}(s) g_j(s) \), here \( G(s) = (G_{ij}(s)) \) is the Gramian matrix of \( g \) and \( (G^{ij}(s)) = G^{-1}(s) \), which is well-defined since \( g \) is linearly independent.

The aim of this article is to show concretely the loss of the quantity of information for the noncanonical representation given by (2).

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The information quantity $I(X, Y)$ of Gaussian random variables $X$ and $Y$ is defined as

$$I(X, Y) = -\frac{1}{2} \log \left\{ 1 - \rho(X, Y)^2 \right\},$$

where $\rho$ is a correlation coefficient.(see [5])

The information quantity $I_1$ on $H_s(B)$ of $X(t), s < t$, canonically represented as (1) is

$$I_1 = \sup \{ I(X(t), Y); Y \in H_s(B) \} = -\frac{1}{2} \log \left\{ 1 - \frac{\int_s^t F(t, u)^2 du}{\int_0^t F(t, u)^2 du} \right\}, \quad (5)$$

since $H_s(B) = \{ \int_0^s \varphi(u) dB(u); \varphi \in L^2[0, s] \}$.

**Theorem.** Let $I_2$ be the information quantity on $H_s(B)$ of $\bar{X}(t), s < t$, represented as (2) with the kernel (4). Then

$$I_2 = -\frac{1}{2} \log \left\{ 1 - \frac{\int_s^t F(t, u)^2 du + \sum_{i,j=1}^N \int_s^t F(t, u) f_i(u) du G_{ij}(s) \int_s^t F(t, u) f_j(u) du}{\int_0^t F(t, u)^2 du} \right\}. \quad (6)$$

**Proof:** By the same reasoning as (5),

$$I_2 = \sup \{ I(\bar{X}(t), Y); Y \in H_s(B) \} = -\frac{1}{2} \log \left\{ 1 - \frac{\int_0^s \bar{F}(t, u)^2 du}{\int_0^t \bar{F}(t, u)^2 du} \right\}. \quad (7)$$

Here we note $\int_0^t F(t, u)^2 du = \int_0^t \bar{F}(t, u)^2 du$ but $\int_0^s F(t, u)^2 du \neq \int_0^s \bar{F}(t, u)^2 du$. Then

$$I_2 = -\frac{1}{2} \log \left\{ 1 - \frac{\int_0^s \bar{F}(t, u)^2 du}{\int_0^t \bar{F}(t, u)^2 du} \right\}, \quad (8)$$

thus we only execute the reduction of $\int_0^s \bar{F}(t, u)^2 du$ substituted (4). After a tedious calculation, (6) can be proven, but the details are omitted here. 

**Remark:** Clearly $G(s)$ is positive definite, thus the inequality $I_2 > I_1$ always holds. This shows that $H_s(B)$ has more information than $H_s(\bar{X})$.

**ACKNOWLEDGEMENT.** The theme of the present article is inspired by Htay’s work [4].
References


