VOLTERRA REPRESENTATIONS OF GAUSSIAN PROCESSES WITH AN INFINITE-DIMENSIONAL ORTHOGONAL COMPLEMENT

YUJI HIBINO
Faculty of Science and Engineering, Saga University, 840-8502, Saga, JAPAN
e-mail: hibinoy@cc.saga-u.ac.jp

HIROSHI MURAOKA
Research Center of Computational Mechanics, Inc., 142-0041, Tokyo, JAPAN
e-mail: muraoka@rccm.co.jp

We consider whether the noncanonical Volterra representation may have an infinite-dimensional orthogonal complement or not by the use of the method of the stationary processes.

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1. Introduction

It is easy to see that, for a given Brownian motion \( B = \{ B(t); t \geq 0 \} \), a Gaussian process \( \tilde{B}_q = \{ \tilde{B}_q(t); t \geq 0 \} \) defined by

\[
\tilde{B}_q(t) = \int_0^t \left( \frac{2q + 1}{q} \frac{u^q}{t^q} - \frac{q + 1}{q} \right) dB(u)
\]

is again a Brownian motion for any nonzero \( q > -1/2 \). Moreover, P. Lévy \cite{levy} pointed out that there were infinitely many polynomials \( P \) so that

\[
\tilde{B}(t) = \int_0^t P(u/t) dB(u)
\]
is again a Brownian motion. Among these many representations of a Brownian motion, there is the only one special representation: so-called a canonical representation [5].

For a Gaussian process \( X = \{X(t); t \geq 0\} \) defined by

\[
X(t) = \int_0^t F(t, u)dB(u),
\]

the representation (2) is said to be canonical with respect to \( B \), if \( B_t(X) = B_t(B) \) for each \( t \), where \( B_t(X) \) is the \( \sigma \)-field generated by \( \{X(s); s \leq t\} \).

As \( B_t(X) \) can be interpreted as past information of \( X \), we can say that in canonical representation theory past information of \( B \) can be completely acquired by that of \( X \). It is noted that, in a Gaussian case, \( B_t(X) = B_t(B) \) is equivalent to \( H_t(X) = H_t(B) \), where \( H_t(X) \) is a closed linear hull of \( \{X(s); s \leq t\} \).

For example, the representation (1) is noncanonical since \( B_q \) satisfies the property

\[
H_t(B) = H_t(\tilde{B}_q) \oplus LS \left\{ \int_0^t u^q dB(u) \right\},
\]

where \( LS\{\ldots\} \) is a linear span of \( \{\ldots\} \).

Let \( g_1, g_2, \ldots, g_N \in L^2_{loc}(0, \infty) \) be linearly independent in \( (0, t) \) for each \( t > 0 \). In the joint work [3], the authors have found how to construct the noncanonical representation of a Brownian motion having the \( N \)-dimensional orthogonal complement whose basis is \( g = \{g_1, g_2, \ldots, g_N\} \):

\[
\tilde{B}_g(t) = \int_0^t \left( 1 - \int_0^t \sum_{i,j=1}^N g_i(s) \Gamma^{ij}(s) g_j(u) ds \right) dB(u)
\]

\[
= B(t) - \int_0^t \int_0^s \sum_{i,j=1}^N g_i(s) \Gamma^{ij}(s) g_j(u) dB(u) ds
\]

is a Brownian motion, and is noncanonical with respect to \( B \) satisfying

\[
H_t(B) = H_t(\tilde{B}_g) \oplus LS \left\{ \int_0^t g_j(u) dB(u), j = 1, 2, \ldots, N \right\},
\]

where

\[
(\Gamma^{ij}(t)) = \Gamma(t)^{-1} = \left( \int_0^t g_i(u) g_j(u) du \right)^{-1}.
\]

We remark that the Grammian matrix \( \Gamma(t) \) is invertible since \( g \) is linearly independent in \((0, t)\).
The form (3) is a Volterra representation. (This terminology is due to [2].) Volterra representations are closely related to the famous innovation theorem, and so on. (see e.g. Hida and Hitsuda [6])

In this article, we shall consider whether the noncanonical Volterra representation may have an infinite-dimensional orthogonal complement by the use of the knowledge of the stationary processes.

In Section 2, we review canonical representation theory for stationary processes, and then we give the result in [4] that there exists a noncanonical representation having an infinite-dimensional orthogonal complement.

In Section 3, we discuss various conditions for Volterra representation to have an infinite-dimensional orthogonal complement.

2. Stationary processes

Suppose that $F(\cdot, \cdot)$ in the representation (2) is a homogeneous function of degree $\alpha$ i.e. $F(at, au) = a^\alpha F(t, u)$ for any positive $a$. By using the transformation

$$Y(s) = \frac{1}{\sqrt{2\pi}} e^{(2\alpha+1)s} X(e^{2s}), \quad s \in \mathbb{R},$$

the process $X$ is transformed into the stationary process

$$Y(s) = \int_{-\infty}^{s} F(1, e^{-2(s-u)}) e^{-(s-u)} dW(u), \quad s \in \mathbb{R},$$

where $dW(u) = \frac{1}{\sqrt{2\pi}} e^{-u} dB(e^{2u})$ is a Wiener measure.

In these representations, each canonical property corresponds as follows: $X$ is canonical with respect to $B$, if and only if $Y$ is canonical with respect to $W$; on the other hand, $X$ is noncanonical with respect to $B$ satisfying

$$H_0(X) \perp \int_{0}^{t} u^q dB(u), \quad t > 0,$$

for $q > -1/2$, if and only if $Y$ is noncanonical with respect to $W$ satisfying

$$H_s(Y) \perp \int_{-\infty}^{s} e^{pu} dW(u), \quad s \in \mathbb{R},$$

for $p = 2q + 1 > 0$.

Concerning stationary processes, canonical representation theory is well developed in deep connections with functional analysis. In this section, we shall give a brief review of the theory.

Let a stationary process $Y = \{Y(s); s \in \mathbb{R}\}$ be represented as

$$Y(s) = \int_{-\infty}^{s} G(s - u) dW(u).$$

(5)
Due to the Paley-Wiener theorem [1], the Fourier transform \( \hat{G} \) of \( G \) lives in the Hardy class \( H^2_+ \) in the upper half-plane \( \mathbb{C}_+ = \{ z \in \mathbb{C}; \Im z > 0 \} \) since \( G \) belongs to \( L^2[0, \infty) \).

It is also known that \( c \in H^2_+ \) has a unique decomposition:

\[
c(\lambda) = C c_O(\lambda) c_I(\lambda)
\]

with

\[
c_O(\lambda) = \sqrt{\frac{2\pi}{i}} \exp \left\{ -\frac{1}{2i} \int_{-\infty}^{\infty} \frac{1+w\lambda}{w-\lambda} \log f(w) \, dw \right\},
\]

\[
c_I(\lambda) = \Pi(\lambda) \exp \left\{ \frac{1}{2i} \int_{-\infty}^{\infty} \frac{1+w\lambda}{w-\lambda} d\beta(w) + ih\lambda \right\},
\]

where \( C \) is a constant of a unit modulus, \( \Pi \) is a Blaschke product, \( f \) is a spectral density function of the process \( Y \), \( h \) is a nonnegative constant, and \( \beta \) is a nondecreasing function of bounded variation whose derivative vanishes almost everywhere. Here, \( c_O \) and \( c_I \) are called an outer function and an inner function, respectively. Both are analytic in \( \mathbb{C}_+ \). The outer function never vanishes in \( \mathbb{C}_+ \). All the zero-points there are undertaken by the inner function. Though the representation kernel \( G \) satisfies

\[
\frac{1}{2\pi} |\hat{G}(\lambda)|^2 = f(\lambda), \quad \lambda \in \mathbb{R},
\]

the outer function \( c_O \) also satisfies

\[
\frac{1}{2\pi} |c_O(\lambda)|^2 = f(\lambda), \quad \lambda \in \mathbb{R}.
\]

Thus the inner function has a unit modulus on the real-axis.

Related to canonical representation, the following fact in [8] is important: The representation (5) is canonical with respect to \( W \), if and only if the inner function of \( \hat{G} \) is absent.

This tells us that the outer function determines the law of the process and that the inner function causes the noncanonical property.

**Example 2.1.** For the representation (1) of a Brownian motion, using the transformation (4) we obtain a stationary Brownian motion (Ornstein-Uhlenbeck process) \( Y \) having the representation (5) with

\[
G(t) = \frac{2q + 1}{q} e^{-(2q+1)t} - \frac{q + 1}{q} e^{-t}, \quad t \geq 0.
\]

The Fourier transform \( c \) of \( G \) is

\[
c(\lambda) = \left( \frac{1}{1-i\lambda} \right) \left( \frac{\lambda - ip}{\lambda + ip} \right), \quad \text{where } p = 2q + 1.
\]
The first factor is the outer function, and the second is the inner one. As we have remarked, it is noncanonical satisfying
\[ H_s(Y) \perp \int_{-\infty}^s e^{p_n} dW(u), \quad s \in \mathbb{R}. \]

The zero-points in \( \mathbb{C}_+ \) are related to the orthogonal complement of the noncanonical representation:
\[ H_s(Y) \perp \int_{-\infty}^s e^{-i\lambda u} dW(u) \text{ in } H_s(W), \]
if and only if the Fourier transform of \( G \) in (5) has a zero-point \( \lambda_0 \) in \( \mathbb{C}_+ \).

As the authors have pointed out in [4], there exists a noncanonical representation having an infinite-dimensional orthogonal complement: The stationary process \( Y \) defined by the representation (5) with
\[ \hat{G}(\lambda) = c_O(\lambda) \prod_{n=1}^{\infty} \frac{\lambda - ip_n}{\lambda + ip_n} \cdot \frac{|p_n - 1|}{p_n - 1}, \quad (6) \]
where
\[ \sum_{n=1}^{\infty} \frac{p_n}{1 + p_n^2} < \infty \text{ and } p_n > 0, \quad (7) \]
satisfies
\[ H_s(Y) \perp \left\{ \int_{-\infty}^s e^{p_n u} dW(u); n \in \mathbb{N} \right\} \text{ in } H_s(W), \quad s \in \mathbb{R}. \quad (8) \]

The condition (7) guarantees the convergence of the infinite product in (6).

By using the inverse transform \( X(t) = \sqrt{2t}Y(\log t) \) of (4), we can prove the following theorem.

**Theorem 2.1.** There exists a noncanonical representation of a Gaussian process \( X \) satisfying
\[ H_t(X) \perp \left\{ \int_0^t u^{q_n} dB(u); n \in \mathbb{N} \right\} \text{ in } H_t(B), \quad t > 0, \quad (9) \]
for the sequence \( \{q_n\} = \{(p_n - 1)/2\} \) satisfying (7).

If we take \( c_O(\lambda) = 1/(1 - i\lambda) \) in (6), we can easily see that there exists a noncanonical representation of a Brownian motion having an infinite-dimensional orthogonal complement.
3. Volterra representations

We call a Volterra representation for the representation

\[ X(t) = B(t) - \int_0^t \int_0^s k(s, u) dB(u) ds, \tag{10} \]

where \( k(s, \cdot) \) belongs to \( L^2(0, s) \) for any \( s > 0 \), satisfying

\[ \int_0^t \left( \int_0^s k(s, u)^2 du \right)^{1/2} ds < \infty. \]

It is well-known in [7] that \( X \) is equivalent to \( B \) (i.e. the distributions of \( X \) and of \( B \) are mutually absolutely continuous), if and only if \( X \) admits a Volterra representation with \( k \in L^2((0, t)^2) \) for any \( t > 0 \), namely,

\[ \int_0^t \int_0^s k(s, u)^2 duds < \infty \text{ for any } t > 0. \]

In this case, the representation (10) is canonical with respect to \( B \).

As we have seen in Section 1, the noncanonical representation (3) of a Brownian motion is always a Volterra representation. Therefore, the Volterra kernel

\[ k(s, u) = \begin{cases} \sum_{i,j=1}^N g_i(s) G^{ij}(s) g_j(u), & t > s, \\ 0, & t < s, \end{cases} \]

is not square-integrable. Needless to say, we can check it by a direct calculation.

In order that the representation kernel of (10) is a homogeneous function of degree zero, we shall restrict to the case of \( k(s, u) = (1/s)\varphi(u/s) \), where \( \varphi \) belongs to \( L^2(0, 1) \). Then the representation (10) turns to

\[ X(t) = \int_0^t \left( 1 - \int_{u/t}^1 \frac{1}{x} \varphi(x) dx \right) dB(u). \tag{11} \]

By using the transformation (4) the stationary process \( Y \) is obtained as in the form

\[ Y(s) = \int_{-\infty}^s e^{-(s-u)} \left( 1 - \int_{u}^{s-u} e^v \psi(v) dv \right) dW(u). \tag{12} \]

Here we put \( \psi(v) = 2e^{-v}\varphi(e^{-2v}) \in L^2(0, \infty) \), for short.

**Proposition 3.1.** If a stationary process \( Y \) of the form (12) satisfies (8) for \( p_n > 0 \), then \( \sup p_n < \infty \).
Proof. If the property
\[ H_s(Y) \perp \int_{-\infty}^{s} e^{pu} dW(u) \]
is satisfied, then
\[ \int_{-\infty}^{s} e^{-(s-u)} \left( 1 - \int_{0}^{s-u} e^{v \psi(v)} dv \right) e^{pu} du = 0, \]
for any \( s \in \mathbb{R} \). It is reduced to
\[ \int_{0}^{\infty} e^{-pv} \psi(v) dv = 1. \]
By using the Schwarz inequality,
\[ \sqrt{2p} \leq \| \psi \|_{L^2(0,\infty)} < \infty. \]
Thus, we have proved the desired statement.

By noting that zero-points of an inner function accumulate either infinity or zero when they are pure imaginary, it is enough to consider whether zero-points may tend to zero.

**Theorem 3.1.** If a stationary process \( Y \) is of the form (12), then, for any sequence \( 0 < p_1 < p_2 < \ldots \), the process \( Y \) never has the property (8).

**Proof.** That \( Y \) satisfies (8) is equivalent to that \( \hat{G} \) has infinitely many zero-points \( \{ ip_n \} \) in \( \mathbb{C}_+ \). However, \( \hat{G} \) is analytic there. So the cluster points of its zero-points should be on the boundary. Thus the zero-points can tend to only infinity. Nevertheless, the zero-points cannot tend to infinity because of the proposition above. Therefore, the proof is finished.

The following lemma is obvious, because an inner function is analytic [1].

**Lemma 3.1.** If the modulus of an inner function is continuous in the closed upper half-plane \( \overline{\mathbb{C}_+} \), then the cluster point of zero-points there of the inner function is only point at infinity, if any.

**Theorem 3.2.** If a stationary Brownian motion \( Y \) is of the form (12) with \( \psi \in L^1(0,\infty) \), then \( Y \) never satisfies (8).

**Proof.** The Fourier transform of the representation kernel
\[ G(t) = e^{-t} \left( 1 - \int_{0}^{t} e^{v \psi(v)} dv \right), \quad t \geq 0, \]
of (12) is

\[ \hat{G}(\lambda) = \frac{1}{1 - i\lambda} \left( 1 - \hat{\psi}(\lambda) \right). \]

Therefore the inner function of (12) is \( 1 - \hat{\psi}(\lambda) \), since the outer function of a stationary Brownian motion is \( 1/(1 - i\lambda) \). For \( \psi \) belongs to \( L^1(0, \infty) \), its Fourier transform is continuous in \( \mathbb{C}_+ \). By assuming that \( Y \) has the property (8), the zero-points \( \{ ip_n \} \) can tend only to infinity because of the lemma above. However, \( \{ p_n \} \) cannot tend to infinity thanks to Proposition 3.1.

\[ \square \]

Noting that

\[ \int_0^\infty |\psi(v)| dv = \int_0^1 \left| \frac{1}{\sqrt{x}} \varphi(x) \right| dx, \]

we can prove the following theorem.

**Theorem 3.3.** If a Brownian motion \( X \) is of the form (11) with \( \frac{1}{\sqrt{x}} \varphi(x) \in L^1(0,1) \), then \( X \) never satisfies (9).

4. Concluding remark

(I) We have considered only the case where zero-points of an inner function are pure imaginary.

According to Proposition 3.1, the imaginary parts of zero-points of an inner function cannot tend to infinity; on the other hand, according to Theorem 3.2, under some conditions they cannot tend to zero. However, even if we restrict the process \( Y \) of (12) to be real-valued, the inner function may have zero-points in \( \mathbb{C}_+ \) besides pure imaginary. The condition for a stationary process \( Y \) to be real-valued is that \( \hat{G} \) is symmetric with respect to the imaginary-axis i.e. \( \hat{G}(z) = \hat{G}(-z^*) \).

For example, the sequence \( \{ \zeta_n \} = \{ \pm \xi_n + i\eta \} \) can be the zero-points of an inner function. Indeed,

\[ cf(\lambda) = \prod_{n=1}^{\infty} \frac{\lambda - \zeta_n}{\lambda - \zeta_n^*} \cdot \frac{|\zeta_n^2 + 1|}{\zeta_n^2 + 1} \]

can be an inner function if

\[ \sum_{n=1}^{\infty} \frac{\eta}{1 + \xi_n^2 + \eta^2} < \infty \text{ and } \eta > 0. \]
In this case, the orthogonal complement of $H_s(Y)$ in $H_s(W)$ contains

$$LS \left\{ \int_{-\infty}^{s} e^{yu} \cos(\xi_n u) dW(u), \int_{-\infty}^{s} e^{yu} \sin(\xi_n u) dW(u); n \in \mathbb{N} \right\}$$

for $s \in \mathbb{R}$.

(II) Thanks to the innovation theorem, if

$$H_t(X) \perp \int_{0}^{t} k(t, u) dB(u) \text{ in } H_t(B), \quad t > 0,$$

then $X$ of the form (10) is a Brownian motion. Especially if $X$ is of the form (11) with $\varphi(x) = \sum_{n \in \mathbb{N}} a_n x^{q_n}$ and satisfies (9), then $X$ is a Brownian motion.

The condition for $X$ of a Volterra representation (10) to be a Brownian motion is expressed

$$k(t, u) = \int_{0}^{u} k(t, v) k(u, v) dv, \quad \forall t > \forall u > 0.$$ 

For $X$ of the form (11), it is reduced to

$$\varphi(x) = \int_{0}^{1} \varphi(xy) \varphi(y) dy, \quad 0 < \forall x < 1.$$ 

In terms of $\psi$ in (12), this corresponds to

$$\psi(s) = \int_{0}^{\infty} \psi(s + u) \psi(u) du, \quad \forall s > 0.$$ 

Finally we point out that if $\varphi$ is of the form $\varphi(x) = \sum_{n \in \mathbb{N}} a_n x^{q_n}$, the condition is

$$\sum_{n \in \mathbb{N}} \frac{a_n}{q_n + q_m + 1} = 1, \quad \forall m \in \mathbb{N}.$$ 

References