Canonical property of representations of Gaussian processes with singular Volterra kernels

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Abstract
We consider the Gaussian process $X_\lambda$ defined by parameterizing a singular kernel of Volterra-type introduced in [1]. The kernel has a close connection with the noncanonical representation. The result is that the representation is canonical (resp. noncanonical) if $\lambda < 1/2$ (resp. $\lambda > 1/2$), being independent of the choice of $g$ of a class of functions (Theorem 3).

1 Introduction
Let $X = \{X(t); t \in [0,1]\}$ be a centered Gaussian process, which has a representation such as

$$X(t) = \int_0^t F(t,u)dB(u), \quad t \in [0,1],$$

(1)

by the use of a Brownian motion $B = \{B(t); t \in [0,1]\}$. If it is satisfied that $H_t(X) \equiv LS\{X(s); s \leq t\} \equiv$ the linear space spanned by $\{X(s); s \leq$
(t) is equal to $H_t(B)$, for each $t \in [0,1]$, then the representation of $X$ is called canonical. The concept was introduced by Lévy [4] and soon later a systematic theory was developed by Hida [2].

In the previous paper [1], we have constructed a noncanonical representation of a Brownian motion associated with a certain Volterra-type operator $K_g$. Let us explain the circumstance in detail. For any natural number $N$ and for any linearly independent system $g = \{g_1, g_2, \ldots, g_N\}$ in $L^2[0,t]$, $t > 0$, the Gramian matrix $G(t) = \left(\int_0^t g_i(u)g_j(u)du\right)$ is regular for any $t > 0$. Thus we have defined the Volterra integral operator $K_g$ whose integral kernel is

\[
k_g(s,u) = \begin{cases} 
2 \sum_{i,j=1}^N g_i(s)G^{ij}(s)g_j(u), & s \geq u, \\
0, & s < u,
\end{cases}
\]

where $(G^{ij}(t)) = G^{-1}(t)$. The representation

\[
\bar{B}_g(t) = \int_0^t (I - K_g^*)1_{[0,t]}(u)dB(u), \quad t \in [0,1],
\]

is noncanonical and the process $\bar{B}_g$ is a Brownian motion satisfying

\[
H_t(\bar{B}_g) = H_t(B) \ominus LS \left\{ \int_0^t g_i(u)dB(u); i = 1, \ldots, N \right\}
\]

for each $t \in [0,1]$. Lévy’s examples in [5] of noncanonical representations of a Brownian motion are included in the representations above by taking $N = 1$ and $g_1(u) = u^q$ with $q > -1/2$.

In the present article, the authors will provide an additional property of the Volterra operator $K_g$. That is to say, we shall consider the operators $I - \lambda K_g^*$ parameterized by the real number $\lambda$. The main result is Theorem 3, which says that the Gaussian process having the representation

\[
X_{\lambda}(t) = \int_0^t (I - \lambda K_g^*)1_{[0,t]}(u)dB(u),
\]

is canonical (resp. noncanonical) if $\lambda < 1/2$ (resp. $\lambda > 1/2$). In the case of $\lambda = 1/2$, only for the case $N = 1$ the complete result will be given in Section 3.

The key lemma for the proof of the main result may have its interest. Namely, the probability laws of $X_\lambda$ and $X_{1-\lambda}$ are identical and they give a
pair of canonical and noncanonical representations. As a special case \( \lambda = 1 \),
they are evidently a pair of canonical and noncanonical representations of a Brownian motion.

2 Parameterization of the operator

Let us consider the process \( X_\lambda \) represented by (5).

**Proposition 1** For \( \lambda \leq 0 \), the representation (5) of \( X_\lambda \) is canonical.

**Proof:** For any \( t \in [0,1] \), pick up an \( \alpha \) satisfying
\[
\int_0^t (I - \lambda K_g^*)[0,\ell](u)\alpha(u)du = 0, \quad t \in [0,1],
\]
equally,
\[
\int_0^t 1_{[0,\ell]}(u)(I - \lambda K_g)\alpha(u)du = 0, \quad t \in [0,1].
\]
This is reduced to
\[
(I - \lambda K_g)\alpha(s) = 0, \quad s \in [0,t].
\]
For such an \( \alpha \) in \( L^2[0,t] \),
\[
(\alpha,(I - \lambda K_g)^*\alpha)_t = ((I - \lambda K_g)\alpha,\alpha)_t = 0,
\]
where \( (\cdot,\cdot)_t \) stands for the inner product in \( L^2[0,t] \). Thus
\[
(\alpha,\alpha)_t = \lambda(\alpha,K_g^*\alpha)_t.
\]
On the other hand, because the operator \( (I - K_g^*) \) on \( L^2[0,t] \) is an isometric one,
\[
(K_g^*\alpha,K_g^*\alpha)_t = 2(\alpha,K_g^*\alpha)_t.
\]
Therefore
\[
2(\alpha,\alpha)_t = \lambda(K_g^*\alpha,K_g^*\alpha)_t.
\]
This means that, for \( \lambda \leq 0 \), \( \alpha \) must be identically zero. Thanks to the criterion of the canonical representation [3, Theorem 4.4], the statement is proved.

The following lemma is interesting.
Lemma 2 The probability law of $X_{1-\lambda}$ is the same as that of $X_\lambda$.

Proof: Since $(I - K_{\mathbf{g}}^*)$ is an isometric operator, 

$$(I - K_{\mathbf{g}})(I - K_{\mathbf{g}}^*) = I.$$ 

By an easy calculation, we find 

$$\left( I - \lambda K_{\mathbf{g}} \right) \left( I - \lambda K_{\mathbf{g}}^* \right) = \left( I - (1 - \lambda) K_{\mathbf{g}} \right) \left( I - (1 - \lambda) K_{\mathbf{g}}^* \right). \quad (13)$$

Due to (5),

$$E[X_\lambda(t)X_\lambda(s)] = \left( (I - \lambda K_{\mathbf{g}}) 1_{[0,t]} , (I - \lambda K_{\mathbf{g}}^*) 1_{[0,s]} \right)$$

$$= \left( (I - (1 - \lambda) K_{\mathbf{g}}) 1_{[0,t]} , (I - (1 - \lambda) K_{\mathbf{g}}^*) 1_{[0,s]} \right)$$

$$= E[X_{1-\lambda}(t)X_{1-\lambda}(s)].$$

Theorem 3 For $\lambda < 1/2$, the representation (5) of $X_\lambda$ is canonical, and for $\lambda > 1/2$, the representation is noncanonical.

Proof: As it is known that the operator norm $\|K_{\mathbf{g}}\| \leq 2$ (cf.[1, Theorem 1.1]), $X_\lambda$ is clearly canonical for $|\lambda| < 1/2$, because $(I - \lambda K_{\mathbf{g}})\alpha(s) = 0$, for $s < t$, has the trivial solution only. Thanks to Proposition 1, the former part of the statement is proved. The latter part is due to Lemma 2 and uniqueness of the canonical representation stated in [3, Theorem 4.1].

3 Special case : $N = 1$

The last theorem of the previous section leaves out the case of $\lambda = 1/2$. The case of $N = 1$, we will find that the representation (5) of $X_{1/2}$ is canonical for any square integrable function $g$. 4
Theorem 4 Let $X_\lambda$ be defined as (5) by the use of $g = \{g\}$ in (2). Then the representation (5) is noncanonical, if and only if $\lambda > 1/2$.

Proof: For any $t \in [0,1]$, solve the integral equation
\begin{equation}
(I - \lambda K_g)\alpha(s) = 0, \quad s \in [0,t].
\end{equation}
By differentiation, we obtain an ordinary differential equation
\begin{equation}
\frac{\alpha'(s)}{\alpha(s)} = \frac{g'(s)}{g(s)} + (\lambda - 1)\frac{g(s)^2}{\int_0^s g(u)^2 du}.
\end{equation}
Thus the general solution of (14) have the form
\begin{equation}
\alpha(s) = Cg(s) \left(\int_0^s g(u)^2 du\right)^{\lambda-1}, \quad s \in [0,t],
\end{equation}
where $C$ is a constant. By virtue of the criterion for the canonical representation, The representation of $X_\lambda$ is noncanonical if and only if $\alpha$ is square integrable. Because
\begin{equation}
\int_0^t \left\{ g(s) \left(\int_0^s g(u)^2 du\right)^{\lambda-1} \right\}^2 ds = \int_0^{(g,g)_{[0,t]}} x^{2(\lambda-1)} dx,
\end{equation}
(16) belongs to $L^2[0,t]$ if and only if $\lambda > 1/2$. 

Remark. Some observations of the case of $g \equiv 1$ are given by Yor [6, p.8].

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References


